TIME SERIES ECONOMETRICS

UNIVERSITY OF ST. GALLEN

March 4, 2016

Main Assignment: (20%)

Due Date: 22.04.2016

THE RULES: This is a **group project**. Form **a group of 3-4 people (you must form a group)**. The assignment answers can be emailed to me. Put *"TSE assignment answers"* in the subject header of the email. **Due time is 00.00 (12am, midnight)**. You have over 6 weeks to complete it. Also, you must **attach your clean and documented code** to the assignment when you email it to me.

It is your responsibility to contact me if something is not clear or you suspect an error.

A. ARMA Processes

1) Consider the following AR(2) model:

$$x_t = 1.2x_{t-1} - 0.32x_{t-2} + u_t \tag{1}$$

where $u_t \sim \mathrm{WN}(0, 1)$.

(*a*) Find the roots of the process and determine whether it is stationary or not.

(*b*) Find the variance ($\gamma_x(0)$) and the 1^{st} to 3^{rd} autocovariances ($\gamma_x(1), \gamma_x(2)$ and $\gamma_x(3)$) of the process.

(*c*) Find the 1^{st} to 3^{rd} partial autocorrelations of the process.

2) Consider the ARMA process given by:

$$y_t - 0.8y_{t-1} = u_t + 1.6u_{t-1} + 0.48u_{t-2},. \tag{2}$$

where $u_t \sim \mathrm{WN}(0, 1)$.

1) Calculate the roots of this process.

2) Is this process stationary and/or invertible? If it is not invertible, find the invertible representation.

3) Let x_t evolve as the following AR(1) process:

$$x_t = \alpha x_{t-1} + u_t \tag{3}$$

where $u_t \sim \mathrm{WN}(0, \sigma^2)$ and $|\alpha| < 1$. Given the AR(1) structure, the Long-Run Variance (LRV) of x_t is known to be $\frac{\sigma^2}{(1-\alpha)^2}$. Use the general formulation of the LRV of any stationary and ergodic process, defined as

$$\text{LRV} = \gamma(0) + 2\sum_{j=1}^{\infty}\gamma(j) \tag{4}$$

to verify, **from first principles**, that the LRV for the AR(1) in (3) is in fact $\frac{\sigma^2}{(1-\alpha)^2}$.

4) Suppose you have the following *structural time series model* for x_t

$$x_t = \mu_t + \varepsilon_t \tag{5a}$$
$$\mu_t = \alpha\mu_{t-1} + \eta_t + v_t \tag{5b}$$
$$\eta_t = \beta\eta_{t-1} + u_t \tag{5c}$$

where the three shocks u_t, v_t, and ε_t are each $\mathsf{WN}(0,1)$ and are also mutually uncorrelated at all time horizons.

(*a*) What type of ARMA(p,q) model do the relations in (5) imply for x_t? Use time series algebra to prove your answer.

5) Suppose you have the following *quadratic trend model* for x_t

$$x_t = c + \beta u_{t-1} + \delta_2 t^2 + u_t \tag{6}$$

where $u_t \sim \mathsf{WN}(0,1)$ and $|\beta| < 1$.

(*a*) How many times do you need to difference the series so that the trend component is removed? Use time series algebra to prove your results.

(*b*) After you have removed the trend by differencing, what kind of process does your differenced x_t series follow? What is the variance of this differenced x_t series? What problems do you see with this differenced series when trying to model/estimate it.

6) Consider the following MA(∞) model

$$x_t = u_t + bu_{t-1} + b\rho u_{t-2} + b\rho^2 u_{t-3} + b\rho^3 u_{t-4} + \ldots \tag{7}$$

where $u_t \sim \mathsf{WN}(0,1)$ and $|\rho| < 1$.

(*a*) For which value of b is the MA(∞) in (7) an AR(1).

(*b*) Show that for any other values of b found in (a) the process is an ARMA(1,1) and identify the MA and AR parameters. Do we need to place any restrictions on the magnitude of b? Justify your answer.

7) In the forecasting literature, the exponential smoothing model (ESM) is a very popular forecasting model. Let $s_t = E_{t-1}(x_t)$ denote the one-step ahead forecast of x_t given information up to time $(t-1)$. The forecasts from the ESM are constructed as:

$$s_t = \lambda s_{t-1} + (1-\lambda)x_{t-1} \tag{8}$$

which can equivalently be expressed as:

$$s_t = (1-\lambda)\sum_{j=0}^{\infty} \lambda^j x_{t-1-j}. \tag{9}$$

Show that the forecasts from the ARIMA$(0,1,1)$ model

$$x_t - x_{t-1} = u_t - \beta u_{t-1} \tag{10}$$

are equivalent to those from the ESM model in (9), with $\beta = \lambda$, $u_t \sim \text{WN}(0, \sigma^2)$, and $E_{t-1}(x_{t-i}) = x_{t-i}, \forall i = 1, 2, 3, \ldots$

Hint: Find the AR(∞) representation of $\Delta x_t = (x_t - x_{t-1})$, then expand terms and take expectations at time $(t-1)$.

8) Let the following bivariate *"true"* model be given by:

$$\begin{aligned}(1-\alpha L)y_t &= x_t + u_t,\\ x_t &= (1+\beta L)v_t.\end{aligned}$$

The variable y, however, is not observed and is measured indirectly with an error from the relation:

$$\tilde{y}_t = y_t + \eta_t.$$

All the error terms u_t, v_t, and η_t are white noise $\text{WN}(0,1)$ and uncorrelated with each other.

(a) What type of ARMA process do we get for the observed variable $\tilde{y}_t$? Justify your answer.

(b) The following values are given: $\alpha = 0.8, \beta = -0.6$. Calculate the autocorrelation function of the MA term of the ARMA model of $\tilde{y}_t$.

9) Given the AR(2) model process:

$$(1 - 1.3L + 0.6L^2)x_t = u_t. \tag{11}$$

where $u_t \sim \text{WN}(0, \sigma^2)$.

(a) Compute the Yule-Walker equations and find the variance?

(b) Calculate the values of the ACF and PACF for $j = 1, 2, 3$.

10) Let the following ARMA$(3,2)$ process be given by:

$$(1-0.2L)(1-0.7L+0.12L^2)y_t = (1-0.6L+0.08L^2)u_t. \tag{12}$$

where $u_t \sim \text{WN}(0, \sigma^2)$.

(a) Calculate the roots of this process. Is this process stationary and/or invertible?

(b) Calculate the ACF and PACF for $j = 1, 2, 3$.

(*c*) Find the MA(∞) representation.

(*d*) Find the forecasts $\hat{y}_{T+h|T}$ for all h as well as the forecast errors $e_{T+h} = y_{T+h} - \hat{y}_{T+h|T}$.

B. Simulations

1) Simulate $N = 1000$ sequences of the pure random walk process

$$x_t = c_0 + x_{t-1} + u_t, \;\; u_t \sim \text{N}(0,1)^1 \tag{13}$$

of size $T = 100$.[2] Use $x_0 = 0$ as an initial value for x_t and set $c_0 = 0$. Then, estimate the following three models by ordinary least squares (OLS)

$$\begin{aligned} &(1) \quad x_t = \rho x_{t-1} + u_t \\ &(2) \quad x_t = c + \rho x_{t-1} + u_t \\ &(3) \quad x_t = c + \delta t + \rho x_{t-1} + u_t \end{aligned} \tag{14}$$

where t is a time trend, ie., $t = 1, 2, 3, \ldots, T$, and record the OLS estimates of ρ as well as its OLS standard error $se(\hat{\rho})$.

(*a*) Compute the standard OLS $t-$ratio of the null hypothesis that $\rho = 1$ as $(\hat{\rho} - 1)/se(\hat{\rho})$ for each of the N simulated series and find the 6^{th} percentiles for the 3 scenarios (equations) above.

(*b*) Now set $c_0 = 0.8$ in the process that you simulate from in (13) and repeat what you do in (a) above. What do you observed.

2) Simulate $N = 1000$ sequences from the AR(1) process

$$x_t = 0.3 + \rho x_{t-1} + u_t, \;\; u_t \sim \text{N}(0,1) \tag{15}$$

of sample sizes $T = \{50, 100, 200, 500\}$. Use $x_0 = E(x_t) = \frac{c}{1-\rho}$ as an initial value for x_t. Set $\rho = 0.98$. Then, estimate the following model by ordinary least squares (OLS)

$$x_t = c + \rho x_{t-1} + u_t \tag{16}$$

and record the OLS estimates of $[c \; \rho]$ as well as their OLS standard errors $[se(\hat{c}) \; se(\hat{\rho})]$.

(*a*) Compute the standard OLS $t-$ratio of the null hypothesis that $\rho = 0.98$ as $(\hat{\rho} - 0.98)/se(\hat{\rho})$ for each of the N simulated series and find the 5^{th} percentile. Do this for all 4 sample sizes $T = \{50, 100, 200, 500\}$.

(*b*) Repeat the above simulation exercise, but now use $\rho = 0.95$ as the true population value to simulate from. Construct again $t-$ratios as in part (a) above for the null $\rho = 0.95$ and find the 5^{th} percentile for the 4 sample sizes.

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[1] $\text{N}(0,1)$ stands for a normal random variable with mean 0 and variance 1.

[2] You will loose one observation due to having to lag x_t by one period, so that you will effectively only have $T - 1$ observations for estimation.

(*c*) Repeat the above simulation exercise, but now use $\rho = 0.90$ as the true population value to simulate from. Construct again $t-$ratios as in part (a) above for the null $\rho = 0.90$ and find the 5^{th} percentile for the 4 sample sizes.

(*d*) Comment briefly on what you observe.

3) Take the US real GDP series from http://www.danielbuncic.com/data/usgdpc96.xls log the series so that $y_t = \ln(GDP)$ and perform the following computations.

(*a*) Fit an AR(2) model to Δy_t (with a constant in it) and find its permanent and transitory components using the Beveridge-Nelson (BN) decomposition.

(*b*) Use the Hodrick-Prescott (HP) filter to find the permanent and transitory components of y_t.

(*c*) Plot the transitory components from the BN and HP filters in one figure and comment on what you observe.

4) Consider the following miniature VAR(2) macroeconomic model, where m_t is money growth, y_t is output growth and π_t is inflation, taking the form

$$\begin{aligned}
\begin{bmatrix} \pi_t \\ y_t \\ m_t \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} .7 & .1 & 0 \\ 0 & .4 & .1 \\ .9 & 0 & .8 \end{bmatrix} \begin{bmatrix} \pi_{t-1} \\ y_{t-1} \\ m_{t-1} \end{bmatrix} + \begin{bmatrix} -.2 & 0 & 0 \\ 0 & .1 & .1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \pi_{t-2} \\ y_{t-2} \\ m_{t-2} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{bmatrix} \qquad (17) \\
X_t &= C + A_1 X_{t-1} + A_2 X_{t-2} + U_t
\end{aligned}$$

and $U_t \sim \text{MN}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} .25 & .05 & 0 \\ .05 & .1 & 0 \\ 0 & 0 & .81 \end{bmatrix} \right)$.[3]

(*a*) Put the system into companion form and determine if the system is stationary/stable or not.

(*b*) Find the variance of X_t ($\Gamma_X(0)$) and the (unconditional) mean vector of X_t.

(*c*) Use a Cholesky decomposition to factor $\text{Var}(U_t)$ into PP' form, where P is a lower triangular matrix. Then compute and plot the first 20 impulse responses of π_t, y_t and m_t to each of the three shocks $u_{it}, \forall i = 1, 2, 3$.

(*d*) Forecast y_t for 20 periods into the future.

[3] MN stands for a multivariate normal random variable