PROOF OF PROPOSITION 1

USE CONVERGENCE IN MEAN SQUARE AND THEOREM 3 FROM LECTURE I TO SHOW THAT

$$\bar{x}_T \xrightarrow{P} E(x_t) = \mu \qquad (1)$$

WHERE $\bar{x}_T \equiv T^{-1}\sum_{t=1}^{T} x_t$ (ie. SAMPLE MEAN).

TO SHOW (1) NEED TO SHOW

$$\lim_{T\to\infty} E\left[\bar{x}_T - E(\bar{x}_T)\right]^2 = 0 \quad \text{(ie. MEAN SQUARE CONV.)}$$

STEP 1)

FIND $E(\bar{x}_T)$. NOW

$$\begin{aligned} E(\bar{x}_T) &= E\left(T^{-1}\sum_{t=1}^{T} x_t\right) \\ &= T^{-1}\sum_{t=1}^{T} E(x_t) \qquad \text{(WHY?)} \\ &= T^{-1}\left[\sum_{t=1}^{T} \mu\right] \\ &= T^{-1}\left[T\mu\right] \\ &= \mu. \end{aligned}$$

NOTE HERE THAT $\mu = T^{-1}\sum_{t=1}^{T}\mu$!

STEP 2)

FIND $E\left[\{\bar{x}_T - E(\bar{x}_T)\}^2\right] = E\left[(\bar{x}_T-\mu)^2\right] = Var(\bar{x}_T)$.

WE HAVE

$$\begin{aligned} E\left[(\bar{x}_T-\mu)^2\right] &= E\left[\left(T^{-1}\sum_{t=1}^{T} x_t - \mu\right)^2\right] \\ &= E\left[\left(T^{-1}\sum_{t=1}^{T}(x_t-\mu)\right)^2\right] \qquad \text{(INCLUDING } \mu \text{ IN SUM) (WHY?)} \\ &= \underbrace{T^{-2} E\left[\left(\sum_{t=1}^{T}(x_t-\mu)\right)^2\right]}_{\text{NEED TO EXPAND THIS TERM!}} \end{aligned}$$

SO

$$E\left[(\bar{x}_T-\mu)^2\right] = T^{-2} E\Big[\{(x_1-\mu)+(x_2-\mu)+\ldots+(x_T-\mu)\}\times\{(x_1-\mu)+(x_2-\mu)+\ldots+(x_T-\mu)\}\Big]$$

$$\begin{aligned} = T^{-2} E\Big[& (x_1-\mu)\times\{(x_1-\mu)+(x_2-\mu)+\ldots+(x_T-\mu)\} + \\ & (x_2-\mu)\times\{(x_1-\mu)+(x_2-\mu)+\ldots+(x_T-\mu)\} + \\ & \quad \vdots \qquad\qquad\qquad \vdots \\ & (x_T-\mu)\times\{(x_1-\mu)+(x_2-\mu)+\ldots+(x_T-\mu)\} \Big] \end{aligned}$$

TAKING EXPECTATIONS AND NOTING THAT $E[(x_t-\mu)(x_{t-j}-\mu)] = \gamma(j)$ IN OUR NOTATION, WE GET :

$$E\left[(\bar{x}_T-\mu)^2\right] = T^{-2}\Big[\gamma(0) + \gamma(1) + \gamma(2) + \ldots + \gamma(T-1) +$$

$$\gamma(1) + \gamma(0) + \gamma(1) + \ldots + \gamma(T-2) +$$

$$\gamma(2) + \gamma(1) + \gamma(0) + \ldots + \gamma(T-3) +$$

$$\gamma(3) + \gamma(2) + \gamma(1) + \quad + \gamma(T-4) +$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \ddots$$

$$\gamma(T-1) + \gamma(T-2) + \gamma(T-3) + \ldots + \gamma(0) \Big].$$

NOTE THAT THERE WILL BE

$T \times \gamma(0)$ TERMS (VARIANCES)

$2(T-1) \times \gamma(1)$ TERMS

$2(T-2) \times \gamma(2)$ TERMS ... $2 \times \underbrace{(T-(T-2))}_{2} \times \gamma(T-2)$ TERMS, $2 \times \underbrace{(T-(T-1))}_{1} \times \gamma(T-1)$ TERMS.

OR

$2(T-j) \times \gamma(j) \quad j = 0, \ldots, (T-1).$

THIS YIELDS THEN

$$E\left[(\bar{x}_T-\mu)^2\right] = T^{-2}\left[T\gamma(0) + 2\sum_{j=1}^{T-1}(T-j)\gamma(j)\right]$$

$$\Rightarrow TE\left[(\bar{x}_T-\mu)^2\right] = \gamma(0) + 2\sum_{j=1}^{T-1}\left(1-\frac{j}{T}\right)\gamma(j)$$

| BRING T TO OTHER SIDE AND DEVIDE THE SUMMATION TERM.

$$= \gamma(0) + 2\left[\left(1-\frac{1}{T}\right)\gamma(1) + \left(1-\frac{2}{T}\right)\gamma(2) + \left(1-\frac{3}{T}\right)\gamma(3) + \ldots + \underbrace{\left(1-\frac{T-1}{T}\right)}_{\frac{1}{T}\text{ WEIGHT}}\gamma(T-1)\right]$$

AND WE CAN SEE THAT

A) FOR "LARGE" j, $\gamma(j) \to 0$ BECAUSE CORRELATION WILL BECOME WEAKER AND WEAKER FOR MORE DISTANT OBSERVATIONS DUE TO PROCESS BEING COVARIANCE STATIONARY.

B) FOR "SMALL" j, WEIGHT $\left(1-\frac{j}{T}\right) \to 1$ AS $T \to \infty$ [SEE HAMILTON PAGE 187 FOR PROOF].

WE THEN GET THE FOLLOWING AS $T \to \infty$

$$\lim_{T\to\infty} T\underbrace{E\left[(\bar{x}_T-\mu)^2\right]}_{AVAR(\bar{x}_T)\,!} = \overbrace{\gamma(0) + 2\sum_{j=1}^{\infty}\gamma(j)}^{=\,LRV\,!} = \sum_{j=-\infty}^{\infty}\gamma(j) = \psi(1)\sigma^2\psi(1) \qquad (3)$$

[SEE HAMILTON p. 189].

TAKING ABSOLUTE VALUES OF (3) [ON BOTH SIDES] WE GET

$$\left|\lim_{T\to\infty} TE\left[(x_T-\mu)^2\right]\right| = \left|\gamma(0) + 2\sum_{j=1}^{\infty}\gamma(j)\right|$$

$$\lim_{T\to\infty} TE\left[(\bar{x}_T-\mu)^2\right] \leq \gamma(0) + 2\sum_{j=1}^{\infty}|\gamma(j)|$$

[FOLLOWS FROM TRIANGLE EQUALITY $|A+B| \leq |A|+|B|$; $\{\gamma(0), T, VAR(\bar{x}_T)\} > 0$.]

SINCE $\sum_{j=0}^{\infty}|\gamma(j)| < \infty$ [FROM EQ. 66 IN PROPOSITION 1]

WE HAVE
$$\underbrace{E\left[(\bar{x}_T-\mu)^2\right]}_{AVAR(\bar{x}_T)} \leq T^{-1}\left(\gamma(0) + 2\sum_{t=1}^{\infty}|\gamma(j)|\right) \to 0 \quad \text{AS } T\to\infty$$

SO WE GET $\lim_{T\to\infty} AVAR(\bar{x}_T) = 0$ THEREFORE $\bar{x}_T \xrightarrow{m.s.} \mu \Rightarrow \bar{x}_T \xrightarrow{p} \mu$. QED ■