On a Standard Method for Measuring the Natural Rate of Interest

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Motivation

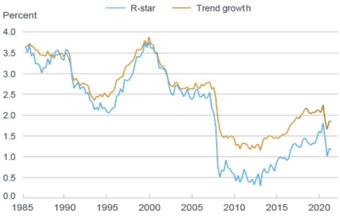
Taylor rule: $i_t = r_t^* + \pi_t + 0.5(\pi_t - \pi^*) + 0.5\tilde{y}_t$ (output gap)

Measuring the Natural Rate of Interest

OVERVIEW | LAUBACH-WILLIAMS ESTIMATES | HOLSTON-LAUBACH-WILLIAMS ESTIMATES | DOWNLOADS

The Laubach-Williams and Holston-Laubach-Williams models provide estimates of the natural rate of interest, or r-star, and related variables. Their approach defines r-star as the real short-term interest rate expected to prevail when an economy is at full strength and inflation is stable.

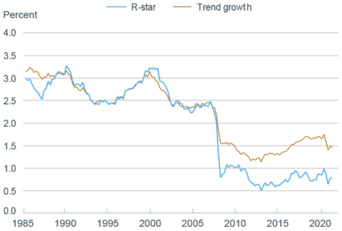
R-STAR FOR THE UNITED STATES
LW Estimation



Source: Laubach and Williams (2003).
Note: We plot estimates of the natural rate of interest (r-star) along with those for the trend growth rate of the U.S. economy, a source of change driving r-star.

The **Laubach-Williams (2003) model** uses data on real GDP, inflation, and the federal funds rate to extract trends in U.S. economic growth and other factors influencing the natural rate of interest. Link through the main navigation tabs above to a data visualization.

R-STAR FOR ADVANCED ECONOMIES
HLW Estimation

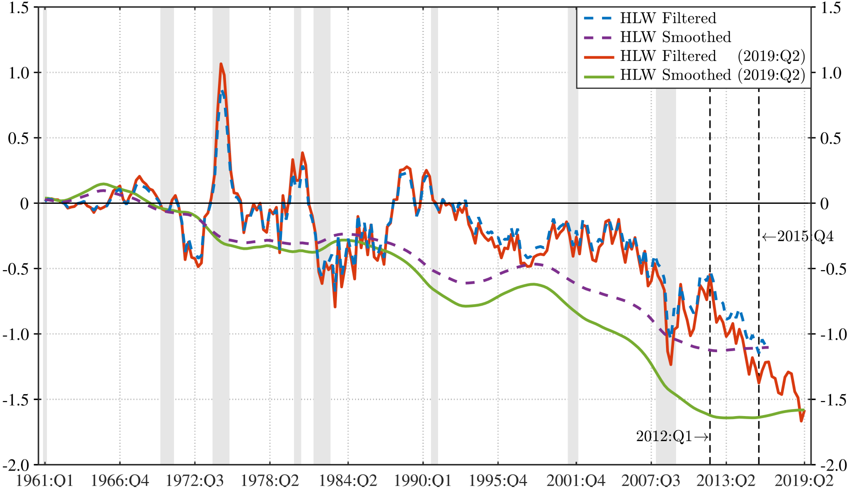


Sources: Holston, Laubach, and Williams (2017); Organisation for Economic Co-operation and Development (OECD).
Notes: Estimates are GDP-weighted averages across the United States, Canada, and the Euro Area. We use OECD estimates of GDP at purchasing power parity. For dates prior to 1995, Euro-Area weights are the summed weights of the eleven original Euro-Area countries.

The **Holston-Laubach-Williams (2017) model** extends this analysis to other advanced economies, estimating r-star and related variables for the United States, Canada, and the Euro Area. Link through the main navigation tabs above to a data visualization.

Natural rate is defined as: $r^* =$ ***trend growth*** g + ***'other factor'*** z

$\Rightarrow$ *'other factor'* z_t driving downward trend in r_t^*



Main results and contribution

Facts about HLW

- size of '*signal to noise*' ratio $\lambda_z = \frac{a_r \sigma_z}{\sigma_{\tilde{y}}}$ drives downward trend in '*other factor*' z_t
- HLW use MUE in "Stage 2" to estimate λ_z due to MLE's '*pile-up at zero*' problems

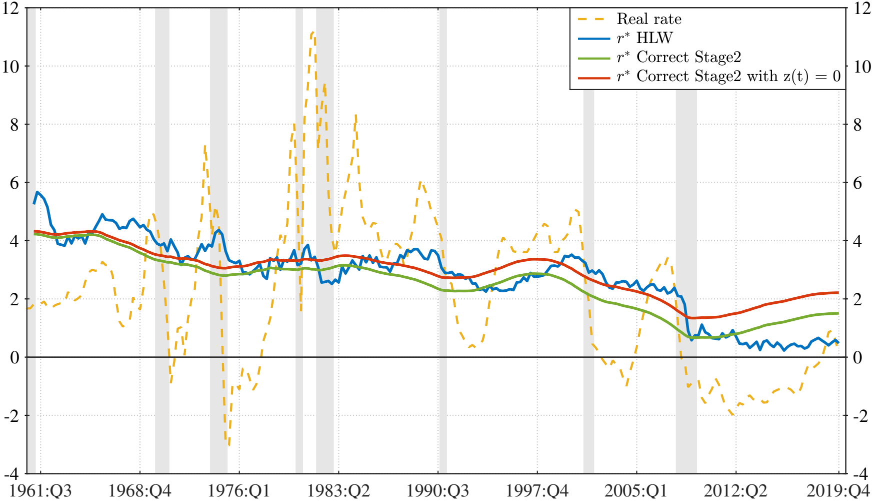
I show that

1) Stage 2 MUE procedure in HLW based on *'unnecessarily'* misspecified model
   $\Rightarrow$ λ_z cannot recover '*signal to noise*' ratio of interest $\frac{a_r \sigma_z}{\sigma_{\tilde{y}}}$
2) HLW modify structural break regressions used in MUE as '*auxilliary*' model
   $\Rightarrow$ results in amplified sequence of $F(\tau)$ structural break statistics

$\Rightarrow$ 1) and 2) together lead to spurious/large estimates of λ_z, trend in z_t, and hence r_t^*

- '*Correct*' Stage 2 MUE of λ_z close to zero and '*highly insignificant'* for US, exactly zero EA, UK, CA

Does it Matter? Real rate and r^* for the US



What is MUE? And why would you use it?

- MUE is fundamentally Indirect Estimation
  - estimate '*auxiliary*' model (or parameters thereof), then back out parameters of interest
- Stock and Watson (1998) introduced MUE for the "... *Estimation of Coefficient Variance in a Time-Varying Parameter Model*" when these are '*small'*
  - when variance '*small*', MLE **can** lead to '*pile-up at zero*' problems
- Stock and Watson (1998) distinguish between **two ML** estimators
  1) **MPLE:** estimates initial state vector with other unknown parameters
  2) **MMLE:** uses '*diffuse prior*', does not estimate initial value
- SW's results show that '*pile-up at zero*' frequencies **very different** for MPLE/MMLE

Stock and Watson (1998): MUE overview

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Cost of MUE

- **Efficiency**! Asymptotic Relative Efficiencies (AREs) deteriorate quickly with increasing values of λ (see Table 2 in SW)
- for $\lambda > 10$, MMLE only needs about 2/3 of data

Estimation of U.S. trend growth per capita

GY_t (400$\Delta\ln(\text{GPD}_t)$ follows Local Level Model (LLM) :

$$GY_t = \beta_t + u_t \tag{1a}$$

$$\Delta\beta_t = (\lambda/T)\eta_t \tag{1b}$$

$$a(L)u_t = \varepsilon_t,\ \eta_t \wedge \varepsilon_t \sim i.i.d.\ N(0,1) \tag{1c}$$

Relation between MUE λ and '*signal-to-noise*' ratio

> *"..., λ is T times the ratio of the long-run standard deviation of $\Delta\beta_t$ to the long run standard deviation of u_t." (SW, p. 351).*

$$\lambda = T\frac{\bar{\sigma}(\Delta\beta_t)}{\bar{\sigma}(u_t)} \Leftrightarrow \frac{\lambda}{T} = \lambda_z = \frac{\sigma_{\Delta\beta}}{\sigma_\varepsilon/a(1)}, \tag{2}$$

Notation:

- $\bar{\sigma}(\cdot) =$ long-run stdev., $\lambda_z = \lambda/T$ as in HLW
- $a(L) = 1 + a_1L + a_2L^2 + a_3L^3 + a_4L^4$ AR(4) lag polynomial, $a\,(1) = 1 + \sum_{i=1}^4 a_i$

MUE of λ from '*auxiliary*' structural break regressions as in SW

1) fit AR(4) to GY_t to remove serial correlation, compute (filtered) $\widetilde{GY}_t = \hat{a}(L)GY_t$
2) test (filtered) $\widetilde{GY}_t$ for break in **unconditional mean** (Chow dummy regressions):

$$\widetilde{GY}_t = \zeta_0 + \zeta_1 D_t(\tau) + \epsilon_t, \tag{3}$$

   $D_t(\tau) = 1$ if $t > \tau$, 0 otherwise, $\tau = \{\tau_0, \tau_0 + 1, \tau_0 + 2, \ldots, \tau_1\}$ grid points
3) construct $\{F(\tau)\}_{\tau=\tau_0}^{\tau_1}$ sequence (F statistic on $\hat{\zeta}_1$ for each τ) and then compute: MW, EW, QLR from $\{F(\tau)\}_{\tau=\tau_0}^{\tau_1}$ and Nyblom's (1989) L statistic from CumSSq
4) use Table 3 look-up values in SW (p. 354) to '*convert*' MW, EW, QLR and Nyblom's (1989) L break statistics into MUE's of λ (λ_z)
5) given $\sigma_{\Delta\beta} = \hat{\lambda}_z \sigma_\varepsilon / a(1)$ estimate LLM by KF(MPLE) **including initial value of** state vector β_{00} ($\hat{\lambda}_z = \hat{\lambda}/T$, where $\hat{\lambda}$ computed in step 4 above)

Stock and Watson (1998): Empirical example

Table 1: Replicated results of Tables 4 and 5 in Stock and Watson (1998)

Test	Statistic	$p-$value	λ	90% CI	$\sigma_{\Delta\beta}$	90% CI
L	0.2094	0.2500	4.0559	$[0, 19.36]$	0.1303	$[0, 0.62]$
MW	1.1588	0.2850	3.4335	$[0, 18.76]$	0.1103	$[0, 0.60]$
EW	0.6821	0.3250	3.0712	$[0, 17.01]$	0.0987	$[0, 0.54]$
QLR	3.3105	0.4800	0.7786	$[0, 13.26]$	0.0250	$[0, 0.42]$

(times series plot of SW's trend growth estimates and estimates where MUE shrinks to 0)

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Parameter	MPLE	MMLE	MUE(0.13)	MUE(0.62)	SW.GAUSS
$\sigma_{\Delta\beta}$	0	0.04440098	0.13	0.62	0.13
σ_{ε}	3.85199480	3.85859423	3.84661923	3.78210658	3.84661917
AR(1)	0.33708321	0.34025234	0.33501453	0.31544472	0.33501454
AR(2)	0.12890328	0.13074607	0.12742313	0.12015642	0.12742309
AR(3)	−0.00917384	−0.00725108	−0.01017060	−0.01488988	−0.01017052
AR(4)	−0.08564442	−0.08247862	−0.08680297	−0.09156066	−0.08680298
β_{00}	1.79589936	—	2.44099926	2.67150007	2.44099940
Log-likelihood	−539.77274703	−547.48046450	−540.69267706	−544.90718114	−540.69267706

(times series plot of SW's trend growth estimates and estimates where MUE shrinks to 0)

Model set-up

Holston *et al.*'s (2017) full *'structural'* model of natural rate:

$$\text{Inflation:} \quad b_\pi(L)\pi_t = b_y \tilde{y}_{t-1} + \varepsilon_t^\pi \tag{4a}$$

$$\text{Output gap:} \quad a_y(L)\tilde{y}_t = a_r(L)\left[r_t - 4g_t - z_t\right] + \varepsilon_t^{\tilde{y}} \tag{4b}$$

$$\text{Output trend:} \quad y_t^* = y_{t-1}^* + g_{t-1} + \varepsilon_t^{y^*} \tag{4c}$$

$$\text{Trend growth:} \quad g_t = g_{t-1} + \varepsilon_t^g \tag{4d}$$

$$\text{Other factor:} \quad z_t = z_{t-1} + \varepsilon_t^z. \tag{4e}$$

Notation:

- $b_\pi(L) = (1 - b_\pi L - (1-b_\pi)(L^2 + L^3 + L^4))$, $a_y(L) = (1 - a_{y,1}L - a_{y,2}L^2)$, $a_r(L) = \frac{a_r}{2}(L + L^2)$, with $a_r(1) = a_r$, and $\varepsilon_t^\ell \sim N(0, \sigma_\ell^2)$
- $\tilde{y}_t = y_t - y_t^*$ is the output gap
- $r_t^* = 4g_t + z_t$ is the natural rate

Note: MLE based $\hat{\sigma}_g$ is larger than from MUE $\Rightarrow$ Stage 1 in HLW is not needed

Why is $\lambda_z = a_r\sigma_z/\sigma_{\tilde{y}}$ in Stage 2 in HLW?

- assume $\tilde{y}_t, g_t, a_r(L), a_y(L)$ are known
- formulate LLM using output gap equation (4b) and '*other factor*' equation (4e) as in SW's model in (1) :

$$\overbrace{a_y(L)\tilde{y}_t - a_r(L)[r_t - 4g_t]}^{\text{analogue to } GY_t} = \overbrace{-a_r(L)z_t}^{\beta_t \text{ analogue}} + \overbrace{\varepsilon_t^{\tilde{y}}}^{u_t} \tag{5a}$$

$$\underbrace{-a_r(L)\Delta z_t}_{\Delta\beta_t \text{ analogue}} = \underbrace{-a_r(L)\varepsilon_t^z}_{(\lambda/T)\eta_t \text{ analogue}} \tag{5b}$$

- the '*signal-to-noise*' ratio $\lambda_z = \lambda/T$ for (5) corresponding to SW's (2) is then:

$$\lambda_z = \frac{\lambda}{T} = \frac{\bar{\sigma}(\Delta\beta_t)}{\bar{\sigma}(u_t)} \Rightarrow \frac{\bar{\sigma}(-a_r(L)\Delta z_t)}{\bar{\sigma}(\varepsilon_t^{\tilde{y}})} = \frac{a_r(1)\sigma_z}{\sigma_{\tilde{y}}} = \frac{a_r\sigma_z}{\sigma_{\tilde{y}}} \tag{6}$$

Two main problems with Stage 2 MUE Implementation in HLW

1) HLW '*unnecessarily'* define/estimate a misspecified Stage 2 model:

2) HLW **modify** the *'auxiliary model's'* structural break regressions

1) & 2) together spuriously increase MUE of λ_z and hence downward trend in z_t

Holston, Laubach and Williams (2017): Stage 2 Model

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1) HLW '*unnecessarily*' define/estimate a misspecified Stage 2 model:

Correctly specified	**HLW misspecified**	
$a_y(L)\tilde{y}_t = a_r(L)\left[r_t - 4g_t\right] + \mathring{\varepsilon}_t^{\tilde{y}}$	$a_y(L)\tilde{y}_t = a_0 + a_r(L)\, r_t + a_g g_{t-1} + \mathring{\varepsilon}_t^{\tilde{y}}$	(7a)
$y_t^* = y_{t-1}^* + g_{t-1} + \varepsilon_t^{y^*}$	$y_t^* = y_{t-1}^* + g_{t-2} + \mathring{\varepsilon}_t^{y^*},$	(7b)

where the misspecified $\mathring{\varepsilon}_t^{\tilde{y}}$ in (7a) is composed as follows (Stage 2 results):

$$\mathring{\varepsilon}_t^{\tilde{y}} = \overbrace{-a_r(L)z_t + \varepsilon_t^{\tilde{y}}}^{\mathring{\varepsilon}_t^{\tilde{y}} \text{ necessary terms}} \underbrace{- \overbrace{\left[a_0 + a_g g_{t-1} + a_r(L)4g_t\right]}^{\text{unnecessary terms}}}_{\mathring{v}_t^{\tilde{y}}}. \tag{8}$$

2) HLW **modify** the *'auxiliary model's'* structural break regressions

1) & 2) together spuriously increase MUE of λ_z and hence downward trend in z_t

1) HLW's MUE of λ_z cannot recover $a_r\sigma_z/\sigma_{\tilde{y}}$ as claimed

- from LLM formulation of misspecified Stage 2 model we have:

$$\overbrace{a_y(L)\tilde{y}_t - a_0 - a_r(L)r_t - a_g g_{t-1}}^{\text{misspecified analogue to } \widetilde{GY}_t} = -a_r(L)z_t + \mathring{\nu}_t^{\tilde{y}}$$
$$-a_r(L)\Delta z_t = -a_r(L)\varepsilon_t^z.$$

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- ratio $\lambda_z = \lambda/T$ corresponding to (2) for the misspecified model is then:

$$\lambda_z = \frac{\lambda}{T} = \frac{\bar{\sigma}(-a_r(L)\Delta z_t)}{\bar{\sigma}(\mathring{\nu}_t^{\tilde{y}})} = \frac{a_r(1)\sigma_z}{\bar{\sigma}(\mathring{\nu}_t^{\tilde{y}})} = \frac{a_r\sigma_z}{\bar{\sigma}(\mathring{\nu}_t^{\tilde{y}})}. \tag{9}$$

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- denominator in (9) requires the evaluation of LR stdev $\bar{\sigma}(\mathring{\nu}_t^{\tilde{y}})$, upon rewriting:

$$\mathring{\nu}_t^{\tilde{y}} = \varepsilon_t^{\tilde{y}} - \left[a_0 + \tfrac{(a_g+4a_r)}{2}(g_{t-1}+g_{t-2}) + \tfrac{a_g}{2}\varepsilon_{t-1}^g\right], \tag{10}$$

even if $(a_g + 4a_r) = 0$ in (10), this yields $\bar{\sigma}(\mathring{\nu}_t^{\tilde{y}}) = \frac{a_r\sigma_z}{(\sigma_{\tilde{y}}+a_g\sigma_g/2)} \neq \frac{a_r\sigma_z}{\sigma_{\tilde{y}}}$.

2) Modification of '*auxiliary*' model in structural break regressions

- **Misspecified Stage 2 model**
  - **HLW:**

$$\hat{\tilde{y}}_{t|T} = a_0 + a_{y,1}\hat{\tilde{y}}_{t-1|T} + a_{y,2}\hat{\tilde{y}}_{t-2|T} + \tfrac{a_r}{2}(r_{t-1} + r_{t-2}) + a_g \overbrace{\hat{g}_{t-1|T}}^{\sim I(1)} + \zeta_1 D_t(\tau) + \epsilon_t$$

  - **SW:**

$$\overbrace{\left(\hat{a}_y(L)\hat{\tilde{y}}_{t|T} - \hat{a}_0 - \hat{a}_r(L)r_t - \hat{a}_g\hat{g}_{t-1|T}\right)}^{\widetilde{GY}_t} = \zeta_0 + \zeta_1 D_t(\tau) + \epsilon_t$$

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- **Correct Stage 2 model**
  - **HLW:**

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  - **SW:**

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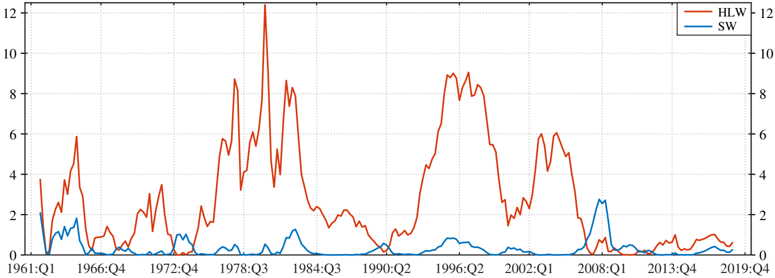
Note:

- Stage 2 parameters $a_{y1,2}$ and a_r **known before** implementing Chow break tests to get MUE of λ_z, there is no need to "*estimate*" them again
- g_{t-1} (and its estimate $\hat{g}_{t-1|T}$) is an $I(1)$ variable, $\Rightarrow$ affects Chow break mapping

Holston, Laubach and Williams (2017): $F(\tau)$ Sequence for the US



(a) HLW's (misspecified) Stage 2 model

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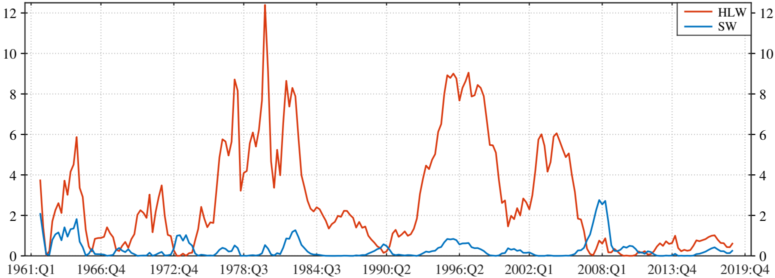

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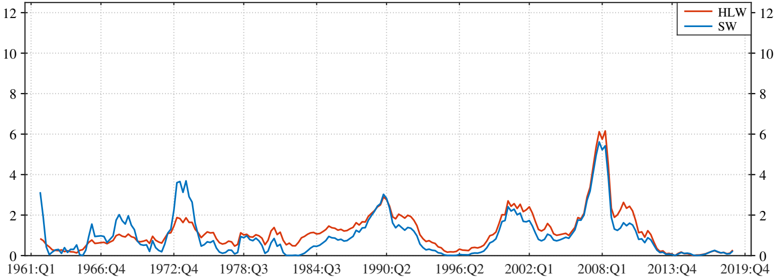

(b) Correct Stage 2 model

Table 2: Stage 2 MUE results of λ_z with corresponding 90% CIs, structural break test statistics and $p-$values

λ_z	HLW's implementation of structural break regressions				
	HLW.R-File	HLW$\left(\hat{\sigma}_g^{\text{MLE}}\right)$	[90% CI]	Correct	[90% CI]
L	—	0.000000	[0, 0.00]	0.012839	[0, 0.07]
MW	0.031855	0.039365	[0, 0.16]	0.015509	[0, 0.07]
EW	0.035415	0.040444	[0, 0.13]	0.015663	[0, 0.07]
QLR	0.044251	0.047740	[0, 0.16]	0.023180	[0, 0.09]
				Corresponding structural break	
L	—	0.037097	(0.9450)	0.170077	(0.3350)
MW	2.747739	3.850795	(0.0150)	1.159557	(0.2850)
EW	2.553645	3.184074	(0.0100)	0.775920	(0.2800)
QLR	12.398151	13.725281	(0.0050)	6.156285	(0.1450)

λ_z other countries

Holston, Laubach and Williams (2017): λ_z and break stats for the US

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λ_z	HLW's implementation of structural break regressions					SW's implementation of structural break regressions					
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L	—	0.000000	[0, 0.00]	0.012839	[0, 0.07]	0.000000	[0, 0.01]	0.000000	[0, 0.00]	0.012839	[0, 0.07]
MW	0.031855	0.039365	[0, 0.16]	0.015509	[0, 0.07]	0.000000	[0, 0.02]	0.000000	[0, 0.01]	0.011947	[0, 0.07]
EW	0.035415	0.040444	[0, 0.13]	0.015663	[0, 0.07]	0.000000	[0, 0.03]	0.000000	[0, 0.01]	0.013230	[0, 0.07]
QLR	0.044251	0.047740	[0, 0.16]	0.023180	[0, 0.09]	0.000000	[0, 0.04]	0.000000	[0, 0.03]	0.020805	[0, 0.08]
Corresponding structural break test statistics ($p-$values in parenthesis)											
L	—	0.037097	(0.9450)	0.170077	(0.3350)	0.049609	(0.8750)	0.037097	(0.9450)	0.170077	(0.3350)
MW	2.747739	3.850795	(0.0150)	1.159557	(0.2850)	0.326527	(0.8150)	0.251819	(0.8900)	0.977255	(0.3550)
EW	2.553645	3.184074	(0.0100)	0.775920	(0.2800)	0.199023	(0.7900)	0.148746	(0.8750)	0.681178	(0.3250)
QLR	12.398151	13.725281	(0.0050)	6.156285	(0.1450)	2.759496	(0.5900)	2.185052	(0.7200)	5.613296	(0.1850)

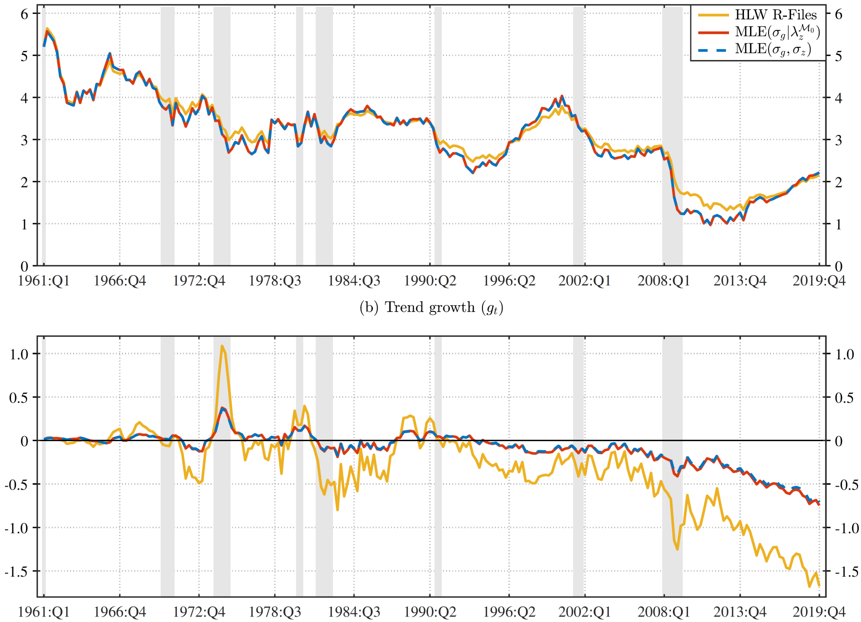
Notes: This table reports the Stage 2 MUEs of λ_z and corresponding structural break statistics computed from the correct Stage 2 model and HLW's (misspecified) Stage 2 model defined in the left and right columns of (6), respectively. The table is split into a left and right block corresponding to the two different implementations of the structural break regressions for each of the Stage 2 models defined in equations (15) to (18). The left block shows HLW's implementation of the structural break regressions. The right block shows SW's implementation. The table is further split into a top and bottom half. The top half shows the MUEs of λ_z. The bottom half lists the corresponding structural break test statistics. The results under ('HLW.R-File') report λ_z estimates obtained from Holston *et al.*'s (2017) R-Files for HLW's (misspecified) Stage 2 model. The ('HLW($\hat{\sigma}_g^{\text{MLE}}$)') column lists estimates of the same model but with σ_g estimated directly by MLE rather than from the first Stage λ_g. Results under the heading ('Correct') are for the correct Stage 2 model, where σ_g is again estimated directly by MLE. The table lists results for all four structural break tests, namely, Nyblom's (1989) L, MW, EW and QLR tests. Values in square brackets in the top part of the table are 90% lower and upper confidence intervals (CIs). Values in parenthesis in the bottom part are $p-$values corresponding to SW's structural break tests. Both, the CIs as well as the $p-$values, were obtained from Stock and Watson's (1998) GAUSS files.

λ_z other countries

Table 3: Stage 3 parameter estimates for the US

θ_3	HLW.R-File	MLE$(\sigma_g \mid \hat{\lambda}_z^{\text{Correct}})$	MLE(σ_g, σ_z)
$a_{y,1}$	1.53991114	1.51796245	1.51804519
$a_{y,2}$	−0.59855575	−0.57839056	−0.57846939
a_r	−0.06786964	−0.06955268	−0.06941225
b_π	0.67083803	0.67182760	0.67187993
b_y	0.07859265	0.07984256	0.07985050
$\sigma_{\tilde{y}}$	0.33378693	0.34501847	0.34535766
σ_π	0.78620285	0.78669831	0.78672365
σ_{y^*}	0.57390968	0.56180867	0.56167781
σ_g (implied)	(0.03073865)	0.04395352	0.04391602
σ_z (implied)	(0.17417263)	(0.06562814)	0.06237003
λ_g (implied)	0.05356007	(0.07823575)	(0.07818721)
λ_z (implied)	0.03541491	0.01323005	(0.01253554)
Log-likelihood	−536.48377160	−535.97760443	−535.97718006

Notes: This table reports the Stage 3 estimates. The first column ('HLW.R-File') gives the estimates from Holston *et al.*'s (2017) R-Files. The second column ('MLE$(\sigma_g \mid \hat{\lambda}_z^{\text{Correct}})$') shows estimates of λ_z from the correct Stage 2 model and SW's implementation of the structural break regressions in (18) (based on the EW structural break test), where σ_g is again estimated by MLE. The last column ('MLE(σ_g, σ_z)') reports estimates where all parameters, including (σ_g, σ_z), are computed directly by MLE. Values in round brackets give the implied (σ_g, σ_z) or (λ_g, λ_z) values constructed from the signal-to-noise ratios $\lambda_g = \sigma_g/\sigma_{y^*}$ and $\lambda_z = a_r\sigma_z/\sigma_{\tilde{y}}$.



(b) Trend growth (g_t)

(c) Other factor (z_t)

Summary

- HLW's MUE implementation cannot recover '*signal to noise*' ratio from Stage 2
- leads to spuriously large estimate of λ_z, exacerbates downward trend in z_t, r_t^*
- $\hat{\lambda}_z$ from correct MUE in Stage 2
  - much smaller for US, '*highly insignificant*'
  - *exactly* 0 for Euro Area, U.K. and Canada
  - smoothed estimates of '*other factor*' z_t zero for all t

Conclusion

Summary

- HLW's MUE implementation cannot recover '*signal to noise*' ratio from Stage 2
- leads to spuriously large estimate of λ_z, exacerbates downward trend in z_t, r_t^*
- $\hat{\lambda}_z$ from correct MUE in Stage 2
  - much smaller for US, '*highly insignificant*'
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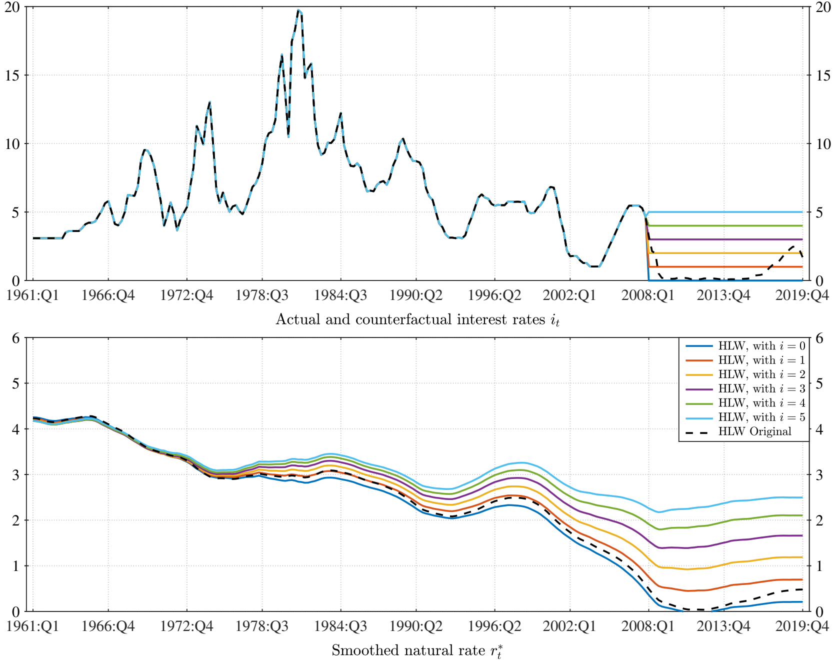
  - smoothed estimates of '*other factor*' z_t zero for all t

Broader issue

- policy rate '*determines*' the natural rate in HLW (and similar models)
- do we really want to '*smooth*' real rates to '*measure*' the natural rate
- recoverability?
  - 5 shocks in HLW, only 2 observables, shock of '*other factor*' ε_t^z not recoverable

Natural rates (smoohted) from HLW at different policy rates



References

Holston, Kathryn, Thomas Laubach and John C. Williams (2017): "Measuring the Natural Rate of Interest: International Trends and Determinants," *Journal of International Economics*, **108**(Supplement 1), S59–S75.

Nyblom, Jukka (1989): "Testing for the Constancy of Parameters over Time," *Journal of the American Statistical Association*, **84**(405), 223–230.

Stock, James H. and Mark W. Watson (1998): "Median Unbiased Estimation of Coefficient Variance in a Time-Varying Parameter Model," *Journal of the American Statistical Association*, **93**(441), 349–358.

Back to 'other factor'

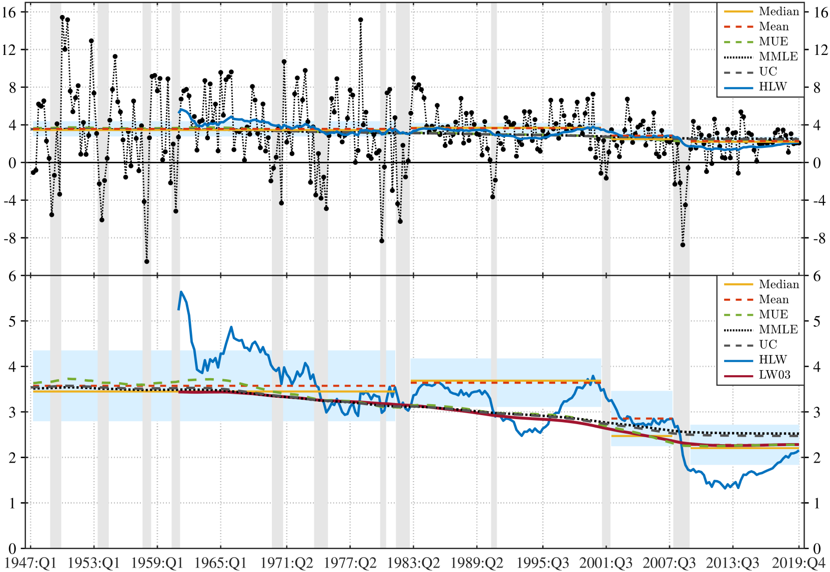


Figure 1: Trend growth estimates

Back to Implementation

Break stats are calculated as:

$$\text{MW} = \frac{1}{N_\tau} \sum_{\tau=\tau_0}^{\tau_1} F(\tau) \tag{11a}$$

$$\text{EW} = \ln \left(\frac{1}{N_\tau} \sum_{\tau=\tau_0}^{\tau_1} \exp \left\{ \frac{1}{2} F(\tau) \right\} \right) \tag{11b}$$

$$\text{QLR} = \max_{\tau \in [\tau_0, \tau_1]} \{F(\tau)\}_{\tau=\tau_0}^{\tau_1}, \tag{11c}$$

where N_τ denotes the number of grid points in τ.
Nyblom's (1989) L statistic computed via sum of squared cumulative sums of $\mathcal{Y}_t$ as:

$$L = T^{-1} \sum_{t=1}^{T} \vartheta_t^2 / \hat{\sigma}_{\mathcal{Y}}^2, \tag{12}$$

where ϑ_t is the scaled cumulative sum of $\tilde{\mathcal{Y}}_t = (\mathcal{Y}_t - \hat{\mu}_{\mathcal{Y}})$, ie., $\vartheta_t = T^{-1/2} \sum_{s=1}^{t} \tilde{\mathcal{Y}}_s$.

▸ Back to Implementation

Table 3. Lookup Table for Constructing Median-Unbiased Estimator of λ for Various Test Statistics When X_t = 1 and D = 1

λ	*L*	*MW*	*EW*	*QLR*	*POI7*	*POI17*
0	.118	.689	.426	3.198	2.693	7.757
1	.127	.757	.476	3.416	2.740	7.825
2	.137	.806	.516	3.594	2.957	8.218
3	.169	1.015	.661	4.106	3.301	8.713
4	.205	1.234	.826	4.848	3.786	9.473
5	.266	1.632	1.111	5.689	4.426	10.354
6	.327	2.018	1.419	6.682	4.961	11.196
7	.387	2.390	1.762	7.626	5.951	12.650
8	.490	3.081	2.355	9.160	6.689	13.839
9	.593	3.699	2.910	10.660	7.699	15.335
10	.670	4.222	3.413	11.841	8.849	16.920
11	.768	4.776	3.868	13.098	10.487	19.201
12	.908	5.767	4.925	15.451	11.598	20.570
13	1.036	6.586	5.684	17.094	13.007	22.944
14	1.214	7.703	6.670	19.423	14.554	24.962
15	1.360	8.683	7.690	21.682	16.153	27.135
16	1.471	9.467	8.477	23.342	18.073	30.030
17	1.576	10.101	9.191	24.920	19.563	32.209
18	1.799	11.639	10.693	28.174	21.662	35.426
19	2.016	13.039	12.024	30.736	24.160	38.465
20	2.127	13.900	13.089	33.313	25.479	40.583
21	2.327	15.214	14.440	36.109	27.687	44.104
22	2.569	16.806	16.191	39.673	30.260	47.239
23	2.785	18.330	17.332	41.955	32.645	50.881
24	2.899	19.020	18.699	45.056	35.011	54.426
25	3.108	20.562	20.464	48.647	37.481	58.172
26	3.278	21.837	21.667	50.983	39.907	60.842
27	3.652	24.350	23.851	55.514	41.146	63.561
28	3.910	26.248	25.538	59.278	43.212	66.782
29	4.015	27.089	26.762	61.311	47.135	71.577
30	4.120	27.758	27.874	64.016	50.134	76.343

NOTE: Entries are the value of the test statistic, for which the value of λ given in the first column is the median-unbiased estimator. Care must be taken to impose the normalization D = 1 when using these estimates of λ. Estimates of τ are computed as λ/T. If the test statistic takes on a value smaller than that in the first row, then the median-unbiased estimate is 0. Estimates for other values of the test statistics can be obtained by interpolation. For example, suppose that QLR = 5.0 is obtained empirically; using linear interpolation, the median unbiased estimator of λ is 4 + (5.0 — 4.848)/(5.689 — 4.848). A software implementation that handles general $\mathbf{X}_t$ for the case D = $\mathbf{I}_k$ is available from the authors by request. All entries in the table were estimated using 5,000 replications and T = 500.

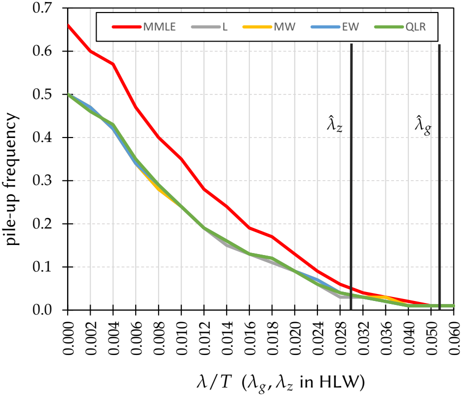
Stock and Watson (1998): Pile-up frequencies

Back to why MUE

Table 1. Pile-Up Probability That $\hat{\lambda}$ = 0 for MLEs and Various Median-Unbiased Estimators

λ	*MPLE*	*MMLE*	*L*	*MW*	*EW*	*QLR*	*POI(7)*	*POI(17)*
0	.96	.66	.50	.50	.50	.50	.50	.50
1	.91	.60	.47	.47	.47	.46	.47	.47
2	.88	.57	.42	.42	.42	.43	.44	.43
3	.81	.47	.34	.34	.34	.35	.35	.37
4	.72	.40	.28	.28	.29	.29	.29	.30
5	.65	.35	.24	.24	.24	.24	.24	.26
6	.56	.28	.19	.19	.19	.19	.18	.20
7	.48	.24	.15	.16	.16	.16	.14	.15
8	.42	.19	.13	.13	.13	.13	.12	.13
9	.37	.17	.11	.12	.12	.12	.09	.10
10	.30	.13	.09	.09	.09	.09	.07	.07
12	.24	.09	.06	.07	.07	.06	.05	.05
14	.15	.06	.03	.04	.04	.04	.03	.02
16	.13	.04	.03	.03	.03	.03	.01	.01
18	.09	.03	.02	.03	.02	.02	.01	.01
20	.07	.02	.01	.01	.01	.01	.01	.01
25	.03	.01	.01	.01	.01	.01	.01	.01
30	.01	.01	.01	.01	.01	.01	.01	.01

NOTE: Entries for MPLE and MMLE for λ = 0 are from Shepard and Harvey (1990). Entries for other values of λ are estimated using 5,000 replications with T = 500. To facilitate the computations, the likelihoods were computed on a discrete grid of 240 equally spaced values of $0 \leq \lambda \leq 60$, and the MLEs were computed by a search over this grid.



- HLW **do not** estimate init.vals of state, use pre-sample data, set $P_{00} = 0.2 \times$identity
- '*signal-to-noise*' ratio estimate **too large** to be consistent with '*pile-up*' problems
- **Stage 1**: $\hat{\lambda}_g = 0.0538$, **Stage 2**: $\hat{\lambda}_z = 0.0302$, a priori, little reason to use MUE

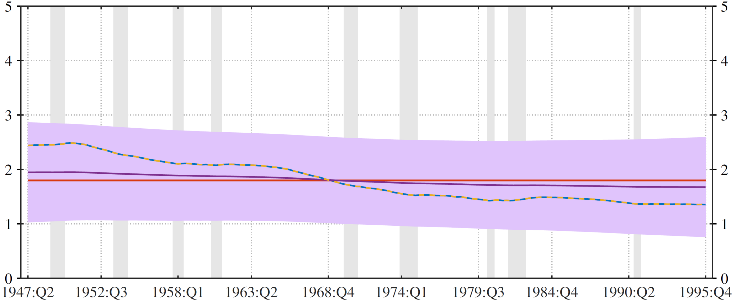
▸ Back to why MUE

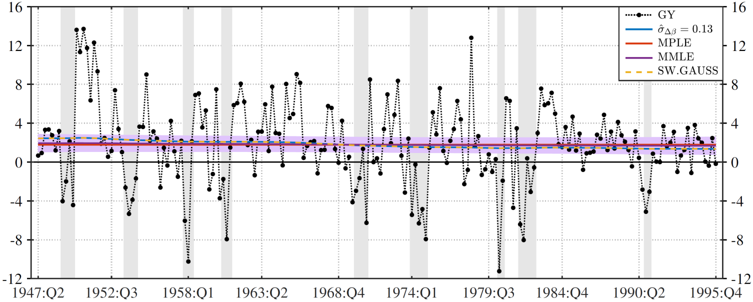
Table 2. Asymptotic Relative Efficiencies of Median-Unbiased Estimators Relative to the MMLE

λ	MPLE	L	MW	EW	QLR	POI(7)	POI(17)
1	.13	1.00	1.00	1.00	1.00	1.07	1.07
2	.19	1.09	1.07	1.07	.96	1.07	1.02
3	.52	1.08	1.10	1.08	1.04	1.02	.94
4	.62	1.06	1.06	1.06	1.02	1.06	1.00
5	.65	.93	.93	.98	.97	1.11	1.14
6	.71	.94	.92	.99	1.00	1.03	1.08
7	.76	.79	.79	.85	.88	.96	1.04
8	.77	.77	.77	.85	.85	.91	.98
9	.75	.69	.69	.75	.77	.86	.89
10	.80	.65	.65	.71	.74	.80	.80
12	.67	.56	.56	.65	.67	.67	.67
14	.57	.50	.49	.57	.57	.57	.57
16	.50	.42	.42	.49	.50	.50	.50
18	.44	.38	.38	.44	.44	.44	.44
20	.40	.33	.33	.40	.40	.40	.40
25	.32	.28	.28	.32	.32	.32	.32
30	.27	.22	.23	.27	.27	.27	.27

NOTE: The reported AREs are the limiting ratio of the number of observations necessary for the MLE to achieve the same probability of being in the region $\tau \pm .5\tau$ as the candidate estimator, as a function of $\lambda = \tau/T$, as described in the text. AREs exceeding 1 indicate greater efficiency than the MLE. Entries are estimates based on interpolating probabilities from the values of λ shown in column 1. These probabilities were estimated using 5,000 replications and $T = 500$ for each value of λ.

▸ Back to SW


5
4
3
2
1
0
1947:Q2
1952:Q3
1958:Q1
1963:Q2
1968:Q4
1974:Q1
1979:Q3
1984:Q4
1990:Q2
1995:Q4


16
12
8
4
0
-4
-8
-12
1947:Q2
1952:Q3
1958:Q1
1963:Q2
1968:Q4
1974:Q1
1979:Q3
1984:Q4
1990:Q2
1995:Q4
GY
$\hat{\sigma}_{\Delta\beta} = 0.13$
MPLE
MMLE
SW.GAUSS

Back to SW

Table 3: Broader replication results of Tables 4 and 5 in Stock and Watson (1998)

Test	Statistic	$p-$value	λ	90% CI	$\sigma_{\Delta\beta}$	90% CI
L	0.0467	0.8950	0.0000	$[0, 4.099]$	0.0000	[0,0.1092]
MW	0.2514	0.8900	0.0000	$[0, 4.296]$	0.0000	[0,0.1145]
EW	0.1321	0.9000	0.0000	$[0, 3.910]$	0.0000	[0,0.1042]
QLR	0.8834	0.9800	0.0000	$[0, 0.000]$	0.0000	$[0, 0.0000]$

Parameter	MPLE	MMLE	MUE($\sigma^L_{\Delta\beta}$)	MUE(CI $-$ $\sigma^L_{\Delta\beta}$)
$\sigma_{\Delta\beta}$	0	0	0	0.10926099
σ_ε	3.86603366	3.87619022	3.86603367	3.87574722
AR(1)	0.31646541	0.32120674	0.31646541	0.32138794
AR(2)	0.14652905	0.14903845	0.14652905	0.14924197
AR(3)	−0.11122061	−0.10873408	−0.11122061	−0.10846721
AR(4)	−0.09512645	−0.09050024	−0.09512645	−0.08983094
β_{00}	2.12011198	—	2.12011200	2.07784473
Log-likelihood	−540.49919714	−548.38308851	−540.49919714	−541.89394940

Notes: This table reports replication results that correspond to Tables 4 and 5 in Stock and Watson (1998) on page 354, but now using real GDP per capita data (2012 chained dollars) obtained from the Federal Reserve Economic Data database (FRED2) with series ID: A939RX0Q048SBEA. The top part of the table shows the 4 different structural break test statistics together with their $p-$values in the first two columns, followed by the corresponding MUE estimates of λ with 90% CIs in square brackets. The last two columns show the implied $\sigma_{\Delta\beta}$ estimate computed from $T^{-1}\lambda \times \sigma_\varepsilon / a(1)$ and 90% CIs in square brackets. The first two columns of the bottom part of the table report results from Maximum Likelihood based estimation, where MPLE estimates the initial value of the state vector β_{00}, while MMLE uses a diffuse prior for the initial value of the state vector with mean zero and the variance set to 10^6. Columns under the heading MUE($\sigma^L_{\Delta\beta}$) and MUE(CI $-$ $\sigma^L_{\Delta\beta}$) show Median Unbiased Estimates when $\sigma_{\Delta\beta}$ is held fixed again at Nyblom's (1989) L test statistic based structural break estimate, respectively, when the upper 90% CI value is used. The row Log-likelihood displays the value of the log-likelihood at the reported parameter estimates. The sample period is the same as in Stock and Watson (1998), that is, from 1947:Q2 to 1995:Q4. The Matlab file `estimate_percapita_trend_growth_v1.m` replicates these results.

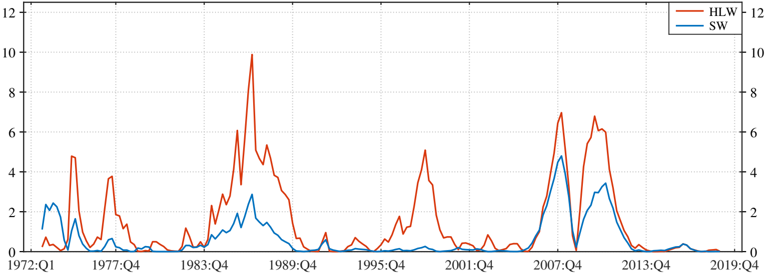
Back to SW

Table 9: Stage 2 parameter estimates

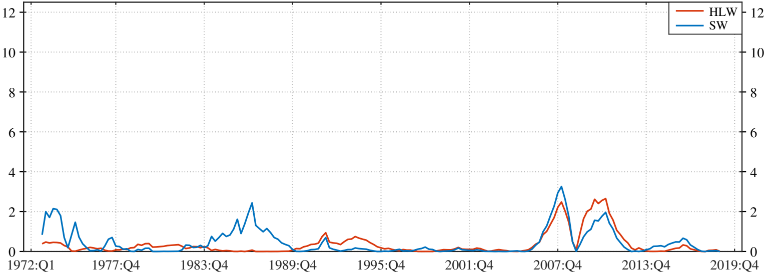
θ_2	HLW.R-File	Replicated	MLE(σ_g)	MLE(σ_g).$\mathcal{M}_0$
$a_{y,1}$	1.5139909	1.5139909	1.4735945	1.4947611
$a_{y,2}$	−0.5709339	−0.5709339	−0.5321668	−0.5531451
a_r	−0.0736647	−0.0736647	−0.0831539	−0.0755563
a_0	−0.2630694	−0.2630694	−0.2548597	—
a_g	0.6078666	0.6078666	0.6277124	—
b_π	0.6627428	0.6627428	0.6655286	0.6692919
b_y	0.0844720	0.0844720	0.0819058	0.0802934
$\sigma_{\tilde{y}}$	0.3582701	0.3582702	0.3636498	0.3742316
σ_π	0.7872280	0.7872280	0.7881906	0.7895137
σ_{y^*}	0.5665698	0.5665698	0.5534537	0.5526273
σ_g (implied)	(0.0305205)	(0.0305205)	0.0437061	0.0448689
λ_g (implied)	0.0538690	0.0538690	(0.0789697)	(0.0811920)
Log-likelihood	−513.5709576	−513.5709576	−513.2849625	−514.1458026

Notes: This table reports replication results for the Stage 2 model parameter vector θ_2 of Holston *et al.* (2017). The first column (HLW.R-File) reports estimates obtained by running Holston *et al.*'s (2017) R-Code for the Stage 2 model. The second column (Replicated) shows the replicated results using the same set-up as in Holston *et al.*'s (2017). The third column (MLE(σ_g)) reports estimates when σ_g is freely estimated by MLE together with the other parameters of the Stage 2 model, rather than imposing the ratio $\lambda_g = \sigma_g/\sigma_{y^*} = 0.0538690378$ obtained from Stage 1. The last column (MLE(σ_g).$\mathcal{M}_0$) provides estimates of the *"correctly specified"* Stage 2 model in (44), with σ_g again estimated directly by MLE. Values in round brackets give the implied σ_g or λ_g values when either λ_g is fixed or when σ_g is estimated. The last row (Log-likelihood) reports the value of the log-likelihood function at these parameter estimates. The Matlab file `Stage2_replication.m` replicates these results.

▸ $F(\tau)$ EA: Back to US



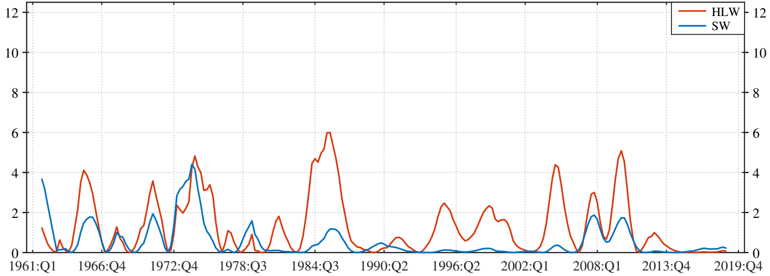
(a) HLW's (misspecified) Stage 2 model (right column block of equation 6)



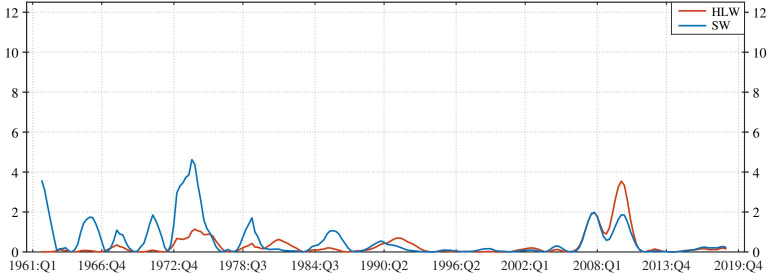
(b) Correct Stage 2 model (left column block in equation 6)

Appendix/Extra slides

▸ $F(\tau)$ UK: Back to US



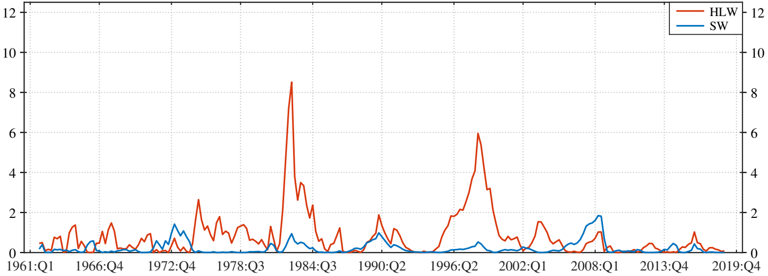
(a) HLW's (misspecified) Stage 2 model (right column block of equation 6)



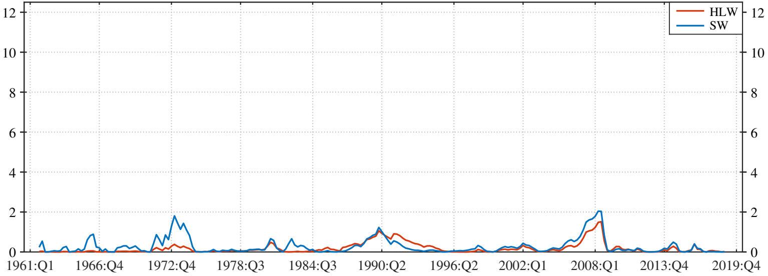
(b) Correct Stage 2 model (left column block in equation 6)

Appendix/Extra slides

▸ $F(\tau)$ CA: Back to US



(a) HLW's (misspecified) Stage 2 model (right column block of equation 6)



(b) Correct Stage 2 model (left column block in equation 6)

λ_z EA: Back to US

Table 5: Stage 2 MUE results of λ_z with corresponding structural break test statistics for the Euro Area

λ_z	HLW's implementation of structural break regressions					SW's implementation of structural break regressions					
	HLW.R-File	HLW$(\hat{\sigma}_g^{\text{MLE}})$	[90% CI]	Correct	[90% CI]	HLW	[90% CI]	HLW$(\hat{\sigma}_g^{\text{MLE}})$	[90% CI]	Correct	[90% CI]
L	—	0.000000	[0, 0.03]	0.000000	[0, 0.03]	0.000000	[0, 0.05]	0.000000	[0, 0.03]	0.000000	[0, 0.03]
MW	0.024235	0.021309	[0, 0.10]	0.000000	[0, 0.03]	0.000000	[0, 0.06]	0.000000	[0, 0.04]	0.000000	[0, 0.04]
EW	0.032869	0.025664	[0, 0.11]	0.000000	[0, 0.04]	0.005006	[0, 0.07]	0.000000	[0, 0.05]	0.000000	[0, 0.05]
QLR	0.044176	0.040711	[0, 0.14]	0.000000	[0, 0.05]	0.020474	[0, 0.09]	0.006325	[0, 0.07]	0.001430	[0, 0.06]
Corresponding structural break test statistics ($p-$values in parenthesis)											
L	—	0.067682	(0.7650)	0.068424	(0.7600)	0.097108	(0.5950)	0.067682	(0.7650)	0.068424	(0.7600)
MW	1.493978	1.270378	(0.2500)	0.351069	(0.7900)	0.622436	(0.5450)	0.454903	(0.6850)	0.455205	(0.6850)
EW	1.525650	1.090316	(0.1750)	0.226251	(0.7500)	0.474062	(0.4550)	0.300121	(0.6400)	0.292027	(0.6500)
QLR	9.882686	8.878605	(0.0450)	2.652280	(0.6150)	4.796794	(0.2650)	3.454149	(0.4500)	3.257853	(0.4900)

Notes: This table reports the Stage 2 MUEs of λ_z and corresponding structural break statistics computed from the correct Stage 2 model and HLW's (misspecified) Stage 2 model defined in the left and right columns of (6), respectively. The table is split into a left and right block corresponding to the two different implementations of the structural break regressions for each of the Stage 2 models defined in equations (15) to (18). The left block shows HLW's implementation of the structural break regressions. The right block shows SW's implementation. The table is further split into a top and bottom half. The top half shows the MUEs of λ_z. The bottom half lists the corresponding structural break test statistics. The results under ('HLW.R-File') report λ_z estimates obtained from Holston *et al.*'s (2017) R-Files for HLW's (misspecified) Stage 2 model. The ('HLW($\hat{\sigma}_g^{\text{MLE}}$)') column lists estimates of the same model but with σ_g estimated directly by MLE rather than from the first Stage λ_g. Results under the heading ('Correct') are for the correct Stage 2 model, where σ_g is again estimated directly by MLE. The table lists results for all four structural break tests, namely, Nyblom's (1989) L, MW, EW and QLR tests. Values in square brackets in the top part of the table are 90% lower and upper confidence intervals (CIs). Values in parenthesis in the bottom part are $p-$values corresponding to SW's structural break tests. Both, the CIs as well as the $p-$values, were obtained from Stock and Watson's (1998) GAUSS files.

▸ λ_z UK: Back to US

Table 8: Stage 2 MUE results of λ_z with corresponding structural break test statistics for the UK

λ_z	HLW's implementation of structural break regressions					SW's implementation of structural break regressions					
	HLW.R-File	HLW$(\hat{\sigma}_g^{\text{MLE}})$	[90% CI]	Correct	[90% CI]	HLW	[90% CI]	HLW$(\hat{\sigma}_g^{\text{MLE}})$	[90% CI]	Correct	[90% CI]
L	—	0.000000	[0, 0.03]	0.000000	[0, 0.03]	0.000000	[0, 0.03]	0.000000	[0, 0.03]	0.000000	[0, 0.03]
MW	0.017600	0.017855	[0, 0.08]	0.000000	[0, 0.02]	0.000000	[0, 0.04]	0.000000	[0, 0.04]	0.000000	[0, 0.04]
EW	0.019307	0.019860	[0, 0.08]	0.000000	[0, 0.03]	0.000000	[0, 0.05]	0.000000	[0, 0.05]	0.000000	[0, 0.05]
QLR	0.022484	0.023628	[0, 0.09]	0.007219	[0, 0.05]	0.014437	[0, 0.07]	0.014828	[0, 0.07]	0.015647	[0, 0.07]
Corresponding structural break test statistics ($p-$values in parenthesis)											
L	—	0.080184	(0.6900)	0.081772	(0.6800)	0.079305	(0.6900)	0.080184	(0.6900)	0.081772	(0.6800)
MW	1.295177	1.319085	(0.2400)	0.309463	(0.8300)	0.540538	(0.6100)	0.546107	(0.6050)	0.555764	(0.6000)
EW	0.984618	1.021807	(0.1900)	0.210029	(0.7750)	0.392172	(0.5350)	0.397701	(0.5300)	0.409986	(0.5150)
QLR	5.993069	6.261263	(0.1400)	3.541260	(0.4350)	4.408113	(0.3050)	4.476559	(0.2950)	4.619930	(0.2800)

Notes: This table reports the Stage 2 MUEs of λ_z and corresponding structural break statistics computed from the correct Stage 2 model and HLW's (misspecified) Stage 2 model defined in the left and right columns of (6), respectively. The table is split into a left and right block corresponding to the two different implementations of the structural break regressions for each of the Stage 2 models defined in equations (15) to (18). The left block shows HLW's implementation of the structural break regressions. The right block shows SW's implementation. The table is further split into a top and bottom half. The top half shows the MUEs of λ_z. The bottom half lists the corresponding structural break test statistics. The results under ('HLW.R-File') report λ_z estimates obtained from Holston *et al.*'s (2017) R-Files for HLW's (misspecified) Stage 2 model. The ('HLW($\hat{\sigma}_g^{\text{MLE}}$)') column lists estimates of the same model but with σ_g estimated directly by MLE rather than from the first Stage λ_g. Results under the heading ('Correct') are for the correct Stage 2 model, where σ_g is again estimated directly by MLE. The table lists results for all four structural break tests, namely, Nyblom's (1989) L, MW, EW and QLR tests. Values in square brackets in the top part of the table are 90% lower and upper confidence intervals (CIs). Values in parenthesis in the bottom part are $p-$values corresponding to SW's structural break tests. Both, the CIs as well as the $p-$values, were obtained from Stock and Watson's (1998) GAUSS files.

▸ λ_z CA: Back to US

Table 11: Stage 2 MUE results of λ_z with corresponding structural break test statistics for Canada

λ_z	HLW's implementation of structural break regressions					SW's implementation of structural break regressions					
	HLW.R-File	HLW$(\hat{\sigma}_g^{\text{MLE}})$	[90% CI]	Correct	[90% CI]	HLW	[90% CI]	HLW$(\hat{\sigma}_g^{\text{MLE}})$	[90% CI]	Correct	[90% CI]
L	—	0.000000	[0, 0.01]	0.000000	[0, 0.01]	0.000000	[0, 0.01]	0.000000	[0, 0.01]	0.000000	[0, 0.01]
MW	0.009032	0.008855	[0, 0.06]	0.000000	[0, 0.01]	0.000000	[0, 0.01]	0.000000	[0, 0.01]	0.000000	[0, 0.02]
EW	0.016314	0.015288	[0, 0.07]	0.000000	[0, 0.01]	0.000000	[0, 0.01]	0.000000	[0, 0.01]	0.000000	[0, 0.02]
QLR	0.032111	0.031447	[0, 0.11]	0.000000	[0, 0.01]	0.000000	[0, 0.02]	0.000000	[0, 0.02]	0.000000	[0, 0.03]
Corresponding structural break test statistics ($p-$values in parenthesis)											
L	—	0.037529	(0.9450)	0.048128	(0.8850)	0.038851	(0.9350)	0.037529	(0.9450)	0.048128	(0.8850)
MW	0.833493	0.824762	(0.4200)	0.182530	(0.9550)	0.221949	(0.9200)	0.214878	(0.9250)	0.278728	(0.8650)
EW	0.801254	0.761333	(0.2900)	0.101482	(0.9500)	0.126872	(0.9100)	0.122280	(0.9200)	0.161823	(0.8550)
QLR	8.513039	8.272703	(0.0550)	1.514397	(0.8850)	1.842522	(0.8050)	1.793354	(0.8200)	2.040637	(0.7600)

Notes: This table reports the Stage 2 MUEs of λ_z and corresponding structural break statistics computed from the correct Stage 2 model and HLW's (misspecified) Stage 2 model defined in the left and right columns of (6), respectively. The table is split into a left and right block corresponding to the two different implementations of the structural break regressions for each of the Stage 2 models defined in equations (15) to (18). The left block shows HLW's implementation of the structural break regressions. The right block shows SW's implementation. The table is further split into a top and bottom half. The top half shows the MUEs of λ_z. The bottom half lists the corresponding structural break test statistics. The results under ('HLW.R-File') report λ_z estimates obtained from Holston *et al.*'s (2017) R-Files for HLW's (misspecified) Stage 2 model. The ('HLW($\hat{\sigma}_g^{\text{MLE}}$)') column lists estimates of the same model but with σ_g estimated directly by MLE rather than from the first Stage λ_g. Results under the heading ('Correct') are for the correct Stage 2 model, where σ_g is again estimated directly by MLE. The table lists results for all four structural break tests, namely, Nyblom's (1989) L, MW, EW and QLR tests. Values in square brackets in the top part of the table are 90% lower and upper confidence intervals (CIs). Values in parenthesis in the bottom part are $p-$values corresponding to SW's structural break tests. Both, the CIs as well as the $p-$values, were obtained from Stock and Watson's (1998) GAUSS files.

Natural rates (filtered) from HLW at different policy rates