

The Rules of Summation

$$\sum_{i=1}^n x_i = x_1 + x_2 + \cdots + x_n$$

$$\sum_{i=1}^n a = na$$

$$\sum_{i=1}^n ax_i = a \sum_{i=1}^n x_i$$

$$\sum_{i=1}^n (x_i + y_i) = \sum_{i=1}^n x_i + \sum_{i=1}^n y_i$$

$$\sum_{i=1}^n (ax_i + by_i) = a \sum_{i=1}^n x_i + b \sum_{i=1}^n y_i$$

$$\sum_{i=1}^n (a + bx_i) = na + b \sum_{i=1}^n x_i$$

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

$$\sum_{i=1}^n (x_i - \bar{x}) = 0$$

$$\begin{aligned} \sum_{i=1}^2 \sum_{j=1}^3 f(x_i, y_j) &= \sum_{i=1}^2 [f(x_i, y_1) + f(x_i, y_2) + f(x_i, y_3)] \\ &= f(x_1, y_1) + f(x_1, y_2) + f(x_1, y_3) \\ &\quad + f(x_2, y_1) + f(x_2, y_2) + f(x_2, y_3) \end{aligned}$$

Expected Values & Variances

$$\begin{aligned} E(X) &= x_1 f(x_1) + x_2 f(x_2) + \cdots + x_n f(x_n) \\ &= \sum_{i=1}^n x_i f(x_i) = \sum_x x f(x) \end{aligned}$$

$$E[g(X)] = \sum_x g(x) f(x)$$

$$\begin{aligned} E[g_1(X) + g_2(X)] &= \sum_x [g_1(x) + g_2(x)] f(x) \\ &= \sum_x g_1(x) f(x) + \sum_x g_2(x) f(x) \\ &= E[g_1(X)] + E[g_2(X)] \end{aligned}$$

$$E(c) = c$$

$$E(cX) = cE(X)$$

$$E(a + cX) = a + cE(X)$$

$$\text{var}(X) = \sigma^2 = E[X - E(X)]^2 = E(X^2) - [E(X)]^2$$

$$\text{var}(a + cX) = E[(a + cX) - E(a + cX)]^2 = c^2 \text{var}(X)$$

Marginal and Conditional Distributions

$$f(x) = \sum_y f(x, y) \quad \text{for each value } X \text{ can take}$$

$$f(y) = \sum_x f(x, y) \quad \text{for each value } Y \text{ can take}$$

$$f(x|y) = P[X = x|Y = y] = \frac{f(x, y)}{f(y)}$$

If X and Y are independent random variables, then $f(x, y) = f(x)f(y)$ for each and every pair of values x and y . The converse is also true.

If X and Y are independent random variables, then the conditional probability density function of X given that

$$Y = y \text{ is } f(x|y) = \frac{f(x, y)}{f(y)} = \frac{f(x)f(y)}{f(y)} = f(x)$$

for each and every pair of values x and y . The converse is also true.

Expectations, Variances & Covariances

$$\begin{aligned} \text{cov}(X, Y) &= E[(X - E(X))(Y - E(Y))] \\ &= \sum_x \sum_y [x - E(X)][y - E(Y)] f(x, y) \end{aligned}$$

$$\rho = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\text{var}(Y)}}$$

$$E(c_1X + c_2Y) = c_1E(X) + c_2E(Y)$$

$$E(X + Y) = E(X) + E(Y)$$

$$\begin{aligned} \text{var}(aX + bY + cZ) &= a^2 \text{var}(X) + b^2 \text{var}(Y) + c^2 \text{var}(Z) \\ &\quad + 2abc \text{cov}(X, Y) + 2acc \text{cov}(X, Z) + 2bcc \text{cov}(Y, Z) \end{aligned}$$

If X , Y , and Z are independent, or uncorrelated, random variables, then the covariance terms are zero and:

$$\begin{aligned} \text{var}(aX + bY + cZ) &= a^2 \text{var}(X) \\ &\quad + b^2 \text{var}(Y) + c^2 \text{var}(Z) \end{aligned}$$

Normal Probabilities

If $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$

If $X \sim N(\mu, \sigma^2)$ and a is a constant, then

$$P(X \geq a) = P\left(Z \geq \frac{a - \mu}{\sigma}\right)$$

If $X \sim N(\mu, \sigma^2)$ and a and b are constants, then

$$P(a \leq X \leq b) = P\left(\frac{a - \mu}{\sigma} \leq Z \leq \frac{b - \mu}{\sigma}\right)$$

Assumptions of the Simple Linear Regression Model

SR1 The value of y , for each value of x , is $y = \beta_1 + \beta_2x + e$

SR2 The average value of the random error e is $E(e) = 0$ since we assume that $E(y) = \beta_1 + \beta_2x$

SR3 The variance of the random error e is $\text{var}(e) = \sigma^2 = \text{var}(y)$

SR4 The covariance between any pair of random errors, e_i and e_j is $\text{cov}(e_i, e_j) = \text{cov}(y_i, y_j) = 0$

SR5 The variable x is not random and must take at least two different values.

SR6 (optional) The values of e are normally distributed about their mean $e \sim N(0, \sigma^2)$

Least Squares Estimation

If b_1 and b_2 are the least squares estimates, then

$$\hat{y}_i = b_1 + b_2x_i$$

$$\hat{e}_i = y_i - \hat{y}_i = y_i - b_1 - b_2x_i$$

The Normal Equations

$$nb_1 + \sum x_i b_2 = \sum y_i$$

$$\sum x_i b_1 + \sum x_i^2 b_2 = \sum x_i y_i$$

Least Squares Estimators

$$b_2 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$$

$$b_1 = \bar{y} - b_2\bar{x}$$

Elasticity

$$\eta = \frac{\text{percentage change in } y}{\text{percentage change in } x} = \frac{\Delta y/y}{\Delta x/x} = \frac{\Delta y}{\Delta x} \cdot \frac{x}{y}$$

$$\eta = \frac{\Delta E(y)/E(y)}{\Delta x/x} = \frac{\Delta E(y)}{\Delta x} \cdot \frac{x}{E(y)} = \beta_2 \cdot \frac{x}{E(y)}$$

Least Squares Expressions Useful for Theory

$$b_2 = \beta_2 + \sum w_i e_i$$

$$w_i = \frac{x_i - \bar{x}}{\sum (x_i - \bar{x})^2}$$

$$\sum w_i = 0, \quad \sum w_i x_i = 1, \quad \sum w_i^2 = 1/\sum (x_i - \bar{x})^2$$

Properties of the Least Squares Estimators

$$\text{var}(b_1) = \sigma^2 \left[\frac{\sum x_i^2}{N \sum (x_i - \bar{x})^2} \right], \quad \text{var}(b_2) = \frac{\sigma^2}{\sum (x_i - \bar{x})^2}$$

$$\text{cov}(b_1, b_2) = \sigma^2 \left[\frac{-\bar{x}}{\sum (x_i - \bar{x})^2} \right]$$

Gauss-Markov Theorem: Under the assumptions SR1–SR5 of the linear regression model the estimators b_1 and b_2 have the *smallest variance of all linear and unbiased estimators* of β_1 and β_2 . They are the **Best Linear Unbiased Estimators** (BLUE) of β_1 and β_2 .

If we make the normality assumption, assumption SR6, about the error term, then the least squares estimators are normally distributed.

$$b_1 \sim N\left(\beta_1, \frac{\sigma^2 \sum x_i^2}{N \sum (x_i - \bar{x})^2}\right), \quad b_2 \sim N\left(\beta_2, \frac{\sigma^2}{\sum (x_i - \bar{x})^2}\right)$$

Estimated Error Variance

$$\hat{\sigma}^2 = \frac{\sum \hat{e}_i^2}{N-2}$$

Estimator Standard Errors

$$\text{se}(b_1) = \sqrt{\text{var}(b_1)}, \quad \text{se}(b_2) = \sqrt{\text{var}(b_2)}$$

t-distribution

If assumptions SR1–SR6 of the simple linear regression model hold, then

$$t = \frac{b_k - \beta_k}{\text{se}(b_k)} \sim t_{(N-2)}, \quad k = 1, 2$$

Interval Estimates

$$P[b_2 - t_c \text{se}(b_2) \leq \beta_2 \leq b_2 + t_c \text{se}(b_2)] = 1 - \alpha$$

Hypothesis Testing

Components of Hypothesis Tests

1. A *null hypothesis*, H_0
2. An *alternative hypothesis*, H_1
3. A *test statistic*
4. A *rejection region*
5. A *conclusion*

If the null hypothesis $H_0 : \beta_2 = c$ is *true*, then

$$t = \frac{b_2 - c}{\text{se}(b_2)} \sim t_{(N-2)}$$

Rejection rule for a two-tail test: If the value of the test statistic falls in the rejection region, either tail of the t -distribution, then we reject the null hypothesis and accept the alternative.

Type I error: The null hypothesis is *true* and we decide to *reject* it.

Type II error: The null hypothesis is *false* and we decide *not* to reject it.

p-value rejection rule: When the p -value of a hypothesis test is *smaller* than the chosen value of α , then the test procedure leads to *rejection* of the null hypothesis.

Prediction

$$y_0 = \beta_1 + \beta_2 x_0 + e_0, \quad \hat{y}_0 = b_1 + b_2 x_0, \quad f = \hat{y}_0 - y_0$$

$$\widehat{\text{var}}(f) = \hat{\sigma}^2 \left[1 + \frac{1}{N} + \frac{(x_0 - \bar{x})^2}{\sum (x_i - \bar{x})^2} \right], \quad \text{se}(f) = \sqrt{\widehat{\text{var}}(f)}$$

A $(1 - \alpha) \times 100\%$ confidence interval, or prediction interval, for y_0

$$\hat{y}_0 \pm t_c \text{se}(f)$$

Goodness of Fit

$$\sum (y_i - \bar{y})^2 = \sum (\hat{y}_i - \bar{y})^2 + \sum \hat{e}_i^2$$

$$SST = SSR + SSE$$

$$R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST} = (\text{corr}(y, \hat{y}))^2$$

Log-Linear Model

$$\ln(y) = \beta_1 + \beta_2 x + e, \quad \widehat{\ln(y)} = b_1 + b_2 x$$

$100 \times \beta_2 \approx \%$ change in y given a one-unit change in x .

$$\hat{y}_n = \exp(b_1 + b_2 x)$$

$$\hat{y}_c = \exp(b_1 + b_2 x) \exp(\hat{\sigma}^2/2)$$

Prediction interval:

$$\exp\left[\widehat{\ln(y)} - t_c \text{se}(f)\right], \quad \exp\left[\widehat{\ln(y)} + t_c \text{se}(f)\right]$$

Generalized goodness-of-fit measure $R_g^2 = (\text{corr}(y, \hat{y}_n))^2$

Assumptions of the Multiple Regression Model

$$\text{MR1} \quad y_i = \beta_1 + \beta_2 x_{i2} + \dots + \beta_K x_{iK} + e_i$$

$$\text{MR2} \quad E(y_i) = \beta_1 + \beta_2 x_{i2} + \dots + \beta_K x_{iK} \Leftrightarrow E(e_i) = 0.$$

$$\text{MR3} \quad \text{var}(y_i) = \text{var}(e_i) = \sigma^2$$

$$\text{MR4} \quad \text{cov}(y_i, y_j) = \text{cov}(e_i, e_j) = 0$$

MR5 The values of x_{ik} are not random and are not exact linear functions of the other explanatory variables.

$$\text{MR6} \quad y_i \sim N[(\beta_1 + \beta_2 x_{i2} + \dots + \beta_K x_{iK}), \sigma^2] \\ \Leftrightarrow e_i \sim N(0, \sigma^2)$$

Least Squares Estimates in MR Model

Least squares estimates b_1, b_2, \dots, b_K minimize

$$S(\beta_1, \beta_2, \dots, \beta_K) = \sum (y_i - \beta_1 - \beta_2 x_{i2} - \dots - \beta_K x_{iK})^2$$

Estimated Error Variance and Estimator Standard Errors

$$\hat{\sigma}^2 = \frac{\sum \hat{e}_i^2}{N-K} \quad \text{se}(b_k) = \sqrt{\widehat{\text{var}}(b_k)}$$

Hypothesis Tests and Interval Estimates for Single Parameters

Use t -distribution $t = \frac{b_k - \beta_k}{\text{se}(b_k)} \sim t_{(N-K)}$

t -test for More than One Parameter

$$H_0 : \beta_2 + c\beta_3 = a$$

When H_0 is true $t = \frac{b_2 + cb_3 - a}{\text{se}(b_2 + cb_3)} \sim t_{(N-K)}$

$$\text{se}(b_2 + cb_3) = \sqrt{\text{var}(b_2) + c^2 \text{var}(b_3) + 2c \times \text{cov}(b_2, b_3)}$$

Joint F -tests

To test J joint hypotheses,

$$F = \frac{(SSE_R - SSE_U)/J}{SSE_U/(N-K)}$$

To test the overall significance of the model the null and alternative hypotheses and F statistic are

$$H_0 : \beta_2 = 0, \beta_3 = 0, \dots, \beta_K = 0$$
$$H_1 : \text{at least one of the } \beta_k \text{ is nonzero}$$

$$F = \frac{(SST - SSE)/(K-1)}{SSE/(N-K)}$$

RESET: A Specification Test

$$y_i = \beta_1 + \beta_2 x_{i2} + \beta_3 x_{i3} + e_i \quad \hat{y}_i = b_1 + b_2 x_{i2} + b_3 x_{i3}$$

$$y_i = \beta_1 + \beta_2 x_{i2} + \beta_3 x_{i3} + \gamma_1 \hat{y}_i^2 + e_i, \quad H_0 : \gamma_1 = 0$$

$$y_i = \beta_1 + \beta_2 x_{i2} + \beta_3 x_{i3} + \gamma_1 \hat{y}_i^2 + \gamma_2 \hat{y}_i^3 + e_i, \quad H_0 : \gamma_1 = \gamma_2 = 0$$

Model Selection

$$\text{AIC} = \ln(SSE/N) + 2K/N$$

$$\text{SC} = \ln(SSE/N) + K \ln(N)/N$$

Collinearity and Omitted Variables

$$y_i = \beta_1 + \beta_2 x_{i2} + \beta_3 x_{i3} + e_i$$

$$\text{var}(b_2) = \frac{\sigma^2}{(1 - r_{23}^2) \sum (x_{i2} - \bar{x}_2)^2}$$

When x_3 is omitted, $\text{bias}(b_2^*) = E(b_2^*) - \beta_2 = \beta_3 \frac{\text{cov}(x_2, x_3)}{\text{var}(x_2)}$

Heteroskedasticity

$$\text{var}(y_i) = \text{var}(e_i) = \sigma_i^2$$

General variance function

$$\sigma_i^2 = \exp(\alpha_1 + \alpha_2 z_{i2} + \dots + \alpha_S z_{iS})$$

Breusch-Pagan and White Tests for $H_0 : \alpha_2 = \alpha_3 = \dots = \alpha_S = 0$

$$\text{When } H_0 \text{ is true } \chi^2 = N \times R^2 \sim \chi_{(S-1)}^2$$

Goldfeld-Quandt test for $H_0 : \sigma_M^2 = \sigma_R^2$ versus $H_1 : \sigma_M^2 \neq \sigma_R^2$

$$\text{When } H_0 \text{ is true } F = \hat{\sigma}_M^2 / \hat{\sigma}_R^2 \sim F_{(N_M - K_M, N_R - K_R)}$$

Transformed model for $\text{var}(e_i) = \sigma_i^2 = \sigma^2 x_i$

$$y_i / \sqrt{x_i} = \beta_1 (1 / \sqrt{x_i}) + \beta_2 (x_i / \sqrt{x_i}) + e_i / \sqrt{x_i}$$

Estimating the variance function

$$\ln(\hat{e}_i^2) = \ln(\sigma_i^2) + v_i = \alpha_1 + \alpha_2 z_{i2} + \dots + \alpha_S z_{iS} + v_i$$

Grouped data

$$\text{var}(e_i) = \sigma_i^2 = \begin{cases} \sigma_M^2 & i = 1, 2, \dots, N_M \\ \sigma_R^2 & i = 1, 2, \dots, N_R \end{cases}$$

Transformed model for feasible generalized least squares

$$y_i / \sqrt{\hat{\sigma}_i} = \beta_1 (1 / \sqrt{\hat{\sigma}_i}) + \beta_2 (x_i / \sqrt{\hat{\sigma}_i}) + e_i / \sqrt{\hat{\sigma}_i}$$

Regression with Stationary Time Series Variables

Finite distributed lag model

$$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \dots + \beta_q x_{t-q} + v_t$$

Correlogram

$$r_k = \frac{\sum (y_t - \bar{y})(y_{t-k} - \bar{y})}{\sum (y_t - \bar{y})^2}$$

$$\text{For } H_0 : \rho_k = 0, \quad z = \sqrt{T} r_k \sim N(0, 1)$$

LM test

$$y_t = \beta_1 + \beta_2 x_t + \rho \hat{e}_{t-1} + \hat{v}_t \quad \text{Test } H_0 : \rho = 0 \text{ with } t\text{-test}$$

$$\hat{e}_t = \gamma_1 + \gamma_2 x_t + \rho \hat{e}_{t-1} + \hat{v}_t \quad \text{Test using } LM = T \times R^2$$

$$\text{AR}(1) \text{ error } y_t = \beta_1 + \beta_2 x_t + e_t \quad e_t = \rho e_{t-1} + v_t$$

Nonlinear least squares estimation

$$y_t = \beta_1 (1 - \rho) + \beta_2 x_t + \rho y_{t-1} - \beta_2 \rho x_{t-1} + v_t$$

ARDL(p, q) model

$$y_t = \delta + \delta_0 x_t + \delta_1 x_{t-1} + \dots + \delta_q x_{t-q} + \theta_1 y_{t-1} + \dots + \theta_p y_{t-p} + v_t$$

AR(p) forecasting model

$$y_t = \delta + \theta_1 y_{t-1} + \theta_2 y_{t-2} + \dots + \theta_p y_{t-p} + v_t$$

Exponential smoothing $\hat{y}_t = \alpha y_{t-1} + (1 - \alpha) \hat{y}_{t-1}$

Multiplier analysis

$$\delta_0 + \delta_1 L + \delta_2 L^2 + \dots + \delta_q L^q = (1 - \theta_1 L - \theta_2 L^2 - \dots - \theta_p L^p) \times (\beta_0 + \beta_1 L + \beta_2 L^2 + \dots)$$

Unit Roots and Cointegration

Unit Root Test for Stationarity: Null hypothesis:

$$H_0 : \gamma = 0$$

Dickey-Fuller Test 1 (no constant and no trend):

$$\Delta y_t = \gamma y_{t-1} + v_t$$

Dickey-Fuller Test 2 (with constant but no trend):

$$\Delta y_t = \alpha + \gamma y_{t-1} + v_t$$

Dickey-Fuller Test 3 (with constant and with trend):

$$\Delta y_t = \alpha + \gamma y_{t-1} + \lambda t + v_t$$

Augmented Dickey-Fuller Tests:

$$\Delta y_t = \alpha + \gamma y_{t-1} + \sum_{s=1}^m \alpha_s \Delta y_{t-s} + v_t$$

Test for cointegration

$$\Delta \hat{e}_t = \gamma \hat{e}_{t-1} + v_t$$

Random walk: $y_t = y_{t-1} + v_t$

Random walk with drift: $y_t = \alpha + y_{t-1} + v_t$

Random walk model with drift and time trend:

$$y_t = \alpha + \delta t + y_{t-1} + v_t$$

Panel Data

Pooled least squares regression

$$y_{it} = \beta_1 + \beta_2 x_{2it} + \beta_3 x_{3it} + e_{it}$$

Cluster robust standard errors $\text{cov}(e_{it}, e_{is}) = \psi_{ts}$

Fixed effects model

$$y_{it} = \beta_{1i} + \beta_2 x_{2it} + \beta_3 x_{3it} + e_{it} \quad \beta_{1i} \text{ not random}$$

$$y_{it} - \bar{y}_i = \beta_2 (x_{2it} - \bar{x}_{2i}) + \beta_3 (x_{3it} - \bar{x}_{3i}) + (e_{it} - \bar{e}_i)$$

Random effects model

$$y_{it} = \beta_{1i} + \beta_2 x_{2it} + \beta_3 x_{3it} + e_{it} \quad \beta_{1i} = \bar{\beta}_1 + u_i \text{ random}$$

$$y_{it} - \alpha \bar{y}_i = \bar{\beta}_1 (1 - \alpha) + \beta_2 (x_{2it} - \alpha \bar{x}_{2i}) + \beta_3 (x_{3it} - \alpha \bar{x}_{3i}) + v_{it}^*$$

$$\alpha = 1 - \sigma_e / \sqrt{T \sigma_u^2 + \sigma_e^2}$$

Hausman test

$$t = (b_{FE,k} - b_{RE,k}) / \left[\text{var}(b_{FE,k}) - \text{var}(b_{RE,k}) \right]^{1/2}$$