Exercise Solutions Lecture Notes 2

1)

$$\left.\begin{aligned} E\left[X_t\right] &= \cos(\omega t)\cdot E[A] + \sin(\omega t)\cdot E[B] = 0 \text{ independent of } t \\ V\left(X_t\right) &= \cos^2(\omega t)\, V(A) + \sin^2(\omega t)\, V(B) = 1 \text{ independent of } t \end{aligned}\right\} E\left[X_t^2\right] = 1 < \infty$$

$$\begin{aligned} Cov\left(X_{t+h}, X_t\right) &= Cov\left(A\cos(\omega(t+h)) + B\sin(\omega(t+h)), A\cos(\omega t) + B\sin(\omega t)\right) \\ &= \cos(\omega(t+h))\cdot\cos(\omega t) + \sin(\omega(t+h))\sin(\omega t) \\ &= \cos(\omega h)\ ,\ h \in \mathbb{Z}\ [\text{because } \cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)] \end{aligned}$$

$$\Longrightarrow \gamma_x(t+h\ ,\ t) = \cos(\omega h)\ ,\ \text{independent of } t$$

$$\Longrightarrow \{X_t\} \text{ stationary with ACVF given by } K(h) = \cos(\omega h)$$

$$\overset{\text{Theorem}}{\Longrightarrow} K(h) \text{ nonnegative definite, even}$$

2)

$$\begin{aligned} Cov\left(X_t, X_t\right) &= V\left(X_t\right) = V\left(Z_t\right) + 0.3^2 V\left(Z_{t-1}\right) + 0.4^2 V\left(Z_{t-2}\right) = 1.25 \\ Cov\left(X_{t+1}, Xt\right) &= Cov\left(Z_{t+1} + 0.3Z_t - 0.4Z_{t-1}, Z_t + 0.3Z_{t-1} - 0.4Z_{t-2}\right) = 0.3 - 0.4\cdot 0.3 = 0.18 \\ Cov\left(X_{t+2}, X_t\right) &= ..... = -0.4 \end{aligned}$$

$$\Longrightarrow \gamma_x(t+h\ ,\ t)\begin{cases} 1.25\ , & h = 0 \\ 0.18\ , & h = \pm 1 \\ -0.4\ , & h = \pm 2 \\ 0\ , & else \end{cases}$$

3)

Let us assume that a stationary solution exists

We already proved that this solution equals $X_t = \sum\limits_{j=0}^{\infty} \phi^j Z_{t-j}$

Now let us consider the difference:

$X_t - \phi_1^{n+1} X_{t-(n+1)} = \sum\limits_{j=0}^{n} \phi_1^j Z_{t-j}$, since

$X_t = \phi_1 X_{t-1} + Z_t = \phi_1\left(\phi_1 X_{t-2} + Z_{t-1}\right) + Z_t = ... = \phi_1^{n+1} X_{t-(n+1)} + \sum\limits_{j=0}^{n} \phi_1^j Z_{t-j}$

The variance of $X_t - \phi_1^{n+1} X_{t-(n+1)}$ is given by

$$V\left(X_t - \phi_1^{n+1} X_{t-(n+1)}\right) = V\left(\sum_{j=0}^{n} \phi_1^j Z_{t-j}\right) = \sum_{j=0}^{n} \phi_1^{2j} V\left(Z_{t-j}\right) \underset{|\phi_1|=1}{\overset{\uparrow}{=}} n, \sigma^2 \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty$$

$\Longrightarrow V\left(X_t\right) = \infty$ conflict: (to stationarity of $\{X_t\}$ thus cannot be stationary)

4)

Homework

5) Let us analyse the autoregressive polynomial $\phi(z) = 1 - 0.75z + 0.125z^2 = (1 - 0.5z)(1 - 0.25z)$

$\rightarrow$ roots of the characteristic equation: $z_1 = 2$ and $z_2 = 4$ $\rightarrow$ outside the unit circle

$\Longrightarrow$ process is stationary and causal

Let us analyse the moving - average polynomial: $\theta(z) = 1 + 1.25z$

$\rightarrow$ root of the characteristic equation: $z_1 = -0.8$ $\rightarrow$ inside the unit circle

$\Longrightarrow$ process is not invertible.

6) *a*) Roots of the AR-polynomial: $\phi(z) = 1 + 0.2z - 0.48z^2 = (1 + 0.8z)(1 - 0.6z) : z_1 = \frac{10}{6}\ ;\ z_2 = -\frac{10}{8}$

$\rightarrow$ roots outside the unit circle $\Longrightarrow$ process is stationary and causal

Roots of the MA-polynomial: $\theta(z) = 1$, no root $\Longrightarrow$ process is invertible

$\rightsquigarrow$ causal representation (MA(∞) representation):

$$\left(1 + 0.2z - 0.48z^2\right)\left(\psi_0 + \psi_1 z + \psi_2^2 z + ...\right) = 1$$

$\Longrightarrow \psi_0 = 1\ ;\ \psi_1 = -0.2\ ;\ \psi_2 = (-0.2) \times (-0.2) + 0.48 = 0.0192\ ;$

$\psi_3 = (-0.2) \times 0.0192 + 0.48 \times (-0.2) = -0.09984$

$\psi_4 = (-0.2) \times (-0.09984) + 0.48 \times 0.0192 = 0.0112128$

$\psi_5 = (-0.2) \times 0.0112128 + 0.48 \times (-0.09984) \tilde{=} -0.007$

b) Roots of the AR-polynomial: $\phi(z) = 1 + 1.9z + 0.88z^2 = (1 - 1.1z)(1 - 0.8z) : z_1 = \frac{10}{8}\ ;\ z_2 = \frac{10}{11}$

$\rightarrow z_2$ lies inside the unit circle $\Longrightarrow$ process is not causal

Roots of the MA-polynomial: $\theta(z) = 1 - 0.4z + 0.04z^2 = (1 - 0.2z)^2\ :\ z_{1,2} = 5$

$\rightarrow$ roots lie outside the unit circle $\Longrightarrow$ process is invertible

c) Roots of the AR-polynomial: $\phi(z) = 1 + 0.6z\ :\ z_1 = -\frac{10}{6}$ $\rightarrow$ outside the unit circle

$\Longrightarrow$ process is stationary and causal

Roots of the MA-polynomial: $\theta(z) = 1 + 1.2z\ :\ z_1 = -\frac{10}{12}$ $\rightarrow$ inside the unit circle

$\Longrightarrow$ process is not invertible

$\rightsquigarrow$ causal representation:

$\psi_0 = 1\ ;\ \psi_1 = (-0.6) + 1.2 = 0.6\ ;\ \psi_2 = (-0.6) \times 0.6 = -0.36\ ;$

$\psi_3 = (-0.6) \times (-0.36) = 0.216\ ;\ \psi_4 = (-0.6) \times 0.216 = -0.1296\ ;\ \psi_5 = (-0.6) \times (-0.1296) = 0.07776$

d) Roots of the AR-polynomial: $\phi(z) = 1 + 1.8z + 0.81z^2 = (1 + 0.9z)^2\ :\ z_{1,2} = -\frac{10}{9}$

$\rightarrow$ roots outside the unit circle $\Longrightarrow$ process is stationary and causal

Roots of the MA-polynomial: $\theta(z) = 1 :$ no root $\Longrightarrow$ process is invertible

$\rightsquigarrow$ causal representation:

$\psi_0 = 1$

$\psi_1 = -1.8$

$\psi_2 = (-1.8) \times (-1.8) - 0.81 \times 1 = 2.43$

$\psi_3 = (-1.8) \times 2.43 - 0.81 \times (-1.8) = -2.916$

$\psi_4 = (-1.8) \times (-2.916) - 0.81 \times 2.43 = 3.2805$

$\psi_5 = (-1.8) \times 3.2805 - 0.81 \times (-2.916) = -3.54294$

7) Homework

8) Let us start by computing the ACVF of the original MA(2) process:

$E[X_t] = \mu$

$V(X_t) = \sigma^2\left(1+\frac{49}{4}+4\right) = \sigma^2\frac{69}{4} = \gamma(0) \qquad \rightsquigarrow \rho(0) = 1$

$Cov(X_t, X_{t-1}) = \sigma^2\left(\frac{7}{2}+\frac{7}{2}\cdot(-2)\right) = -\frac{7}{2}\sigma^2 = \gamma(1) \rightsquigarrow \rho(1) = \frac{\gamma(1)}{\gamma(0)} = -\frac{14}{69}$

$Cov(X_t, X_{t-2}) = \sigma^2(-2) = \gamma(2) \qquad \rightsquigarrow \rho(2) = \frac{\gamma(2)}{\gamma(0)} = -\frac{8}{69}$

$\to \gamma(s) = 0\ ,\ s > 2$

Let us verify whether this process is invertible: $\theta(z) = 1+\frac{7}{2}z - 2z^2 = \left(1-\frac{1}{2}z\right)\cdot(1+4z)$

$\Longrightarrow z_1 = 2\ ,\ z_2 = -\frac{1}{4}$ and z_2 lies inside the unit circle $\Longrightarrow$ process is not invertible

Let us define: $\theta^*(L) = \left(1-\frac{1}{2}L\right)\cdot\left(1+\frac{1}{4}L\right) = 1-\frac{1}{4}L-\frac{1}{8}L^2$ and

$X_t^* = \mu + z_t - \frac{1}{4}z_{t-1} - \frac{1}{8}z_{t-2}\ ,\ \{z_t\} \sim WN\left(0\ ,\ \sigma^{*2}\right)$

Let us compute the ACVF for the new process (which is now invertible):

$V(X_t^*) = \sigma^{*2}\left(1+\frac{1}{16}+\frac{1}{64}\right) = \sigma^{*2}\cdot\frac{69}{64} = \gamma(0)^* \rightsquigarrow \rho^*(0) = 1 = \rho(0)$

$Cov\left(X_t^*, X_{t-1}^*\right) = \sigma^{*2}\left(-\frac{1}{4}+\frac{1}{32}\right) = -\frac{7}{32}\sigma^{*2} = \gamma(1)^* \rightsquigarrow \rho^*(1) = -\frac{14}{69} = \rho(1)$

$Cov\left(X_t^*, X_{t-2}^*\right) = \sigma^{*2}\left(-\frac{1}{8}\right) = \gamma(2)^* \rightsquigarrow \rho^*(2) = -\frac{8}{69} = \rho(2)$

Now, if we want also that $\gamma(0) = \gamma(0)^*$: $\sigma^{*2}\frac{69}{64} = \sigma^2\frac{69}{4} \Longleftrightarrow \sigma^{*2} = \underset{(4^2)}{\underset{\|}{16}}\sigma^2$

$\to$ Note that with such a choice of σ^{*2} : $\gamma(h)^* = \gamma(h)\ \forall h$

9) Let us compute the ACVF of the process:

$V(X_t) = \sigma^2\left(1+\theta_1^2+\theta_2^2\right) = \gamma(0) \rightsquigarrow \rho(0) = 1$

$Cov(X_t, X_{t-1}) = \sigma^2(\theta_1+\theta_1\theta_2) = \gamma(1) \rightsquigarrow \rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2}$

$Cov(X_t, X_{t-2}) = \sigma^2(\theta_2) = \gamma(2) \rightsquigarrow \rho(2) = \frac{\gamma(2)}{\gamma(0)} = \frac{\theta_2}{1+\theta_1^2+\theta_2^2}$

$\rightsquigarrow \gamma(s) = 0\ ,\ s>2 \qquad \rightsquigarrow \rho(s) = 0\ ,\ s > 2$

In the particular case with: $\theta_1 = -\frac{17}{10}\ ,\ \theta_2 = -2$:

$$\gamma(h) = \begin{cases} 7.89\sigma^2 & , \ h = 0 \\ 1.7\sigma^2 & , \ h = \pm 1 \\ -2\sigma^2 & , \ h = \pm 2 \\ 0 & , \ else \end{cases} \quad \text{and} \quad \rho(h) = \begin{cases} 1 & , \ h = 0 \\ \frac{170}{789} & , \ h = \pm 1 \\ -\frac{200}{789} & , \ h = \pm 2 \\ 0 & , \ else \end{cases}$$

Find the roots of the MA-polynomial: $\theta(z) = 1 - \frac{17}{10}z - 2z^2$

$= (1-2.5z)\cdot(1+0.8z)\ :\ z_1 = -\frac{10}{8}$ and $z_2 = \frac{2}{5}$

$\rightsquigarrow z_2$ inside the unit circle $\Rightarrow$ process not invertible

10) *a*) For the stationary of the MA(∞) process $X_t = \sum_{j=0}^{\infty}\theta_j Z_{t-j}$, $\{Z_t\} \sim WN\left(0\ ,\ \sigma^2\right)$

it is enough to show: $\sum_{j=0}^{\infty}|\theta_j|^2 < \infty$

in this case: $\sum_{j=0}^{\infty}\theta_j^2 = 1 + b^2 + b^2\rho^2 + b^2\rho^4 + \ldots\ldots = 1 + b^2\left(\sum_{j=0}^{\infty}\left(\rho^2\right)^j\right) \overset{|\rho|<1!}{\hookrightarrow=} 1 + b^2\cdot\frac{1}{1-\rho^2} < \infty$

Moreover we have: $E[X_t] = 0\ ,\ V(X_t) = \sigma^2\left(1+b^2\cdot\frac{1}{1-\rho^2}\right)$

$E[X_tX_{t-s}] = \sigma^2\cdot b\rho^{s-1}\left(1+b\rho+b\rho^2+\ldots\ldots\right) = \gamma(s)$, depends only on s

b) Note: $X_t = Z_t + b\left(Z_{t-1}+\rho Z_{t-2}+\rho^2 Z_{t-2}+\ldots\ldots\right) = Z_t + \frac{b}{1-\rho L}Z_{t-1}$

$\Longleftrightarrow (1-\rho L)X_t = (1-\rho L)Z_t + bZ_{t-1} = (1-\rho L)Z_t + bLZ_t = (1+(b-\rho)L)Z_t$

The process is AR(1): $\Longrightarrow \quad b = \rho$

When $b \neq \rho$: the process is an ARMA(1,1) with coefficients $\phi_1 = \rho\ ,\ \theta_1 = b - \rho$

11) Homework

12) Recall that for an MA(1) model:

$$\gamma(h) = \begin{cases} \sigma^2\left(1+\theta^2\right) & , \; h = 0 \\ \sigma^2\theta & , \; h = \pm 1 \\ 0 & , \; else \end{cases}$$

From the definition: $\Gamma_2 \cdot \phi_2 = \begin{bmatrix} \sigma^2\left(1+\theta^2\right) & \sigma^2\theta \\ \sigma^2\theta & \sigma^2\left(1+\theta^2\right) \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = \gamma_2 = \begin{bmatrix} \sigma^2\theta \\ 0 \end{bmatrix}$

$$\iff \begin{bmatrix} \underbrace{\begin{matrix} 1+\theta^2 & \theta \\ \theta & 1+\theta^2 \end{matrix}}_{\tilde{\Gamma}_2} \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} \theta \\ 0 \end{bmatrix}$$

Now: $\det \tilde{\Gamma}_2 = \det \begin{bmatrix} 1+\theta^2 & \theta \\ \theta & 1+\theta^2 \end{bmatrix} = \left(1+\theta^2\right)^2 - \theta^2 = 1+\theta^2+\theta^4$

$$\tilde{\Gamma}_2^{-1} = \tfrac{1}{\det \tilde{\Gamma}_2} \times \underbrace{\begin{bmatrix} \left(1+\theta^2\right) & -\theta \\ -\theta & \left(1+\theta^2\right) \end{bmatrix}}_{=\mathsf{adj}(\tilde{\Gamma}_2)}$$

$$\implies \quad \phi_{22} = \tfrac{1}{\det \tilde{\Gamma}_2} \begin{bmatrix} -\theta & \left(1+\theta^2\right) \end{bmatrix} \begin{bmatrix} \theta \\ 0 \end{bmatrix} = \tfrac{-\theta^2}{1+\theta^2+\theta^4}.$$