Exercise Solutions Lecture Notes 1

1) *a)*

$$\left.\begin{array}{l}\left.\begin{array}{l}E[X_t]=a \text{ independent of } t\\ V(X_t)=b^2\times\sigma^2+c^2\times\sigma^2=\left(b^2+c^2\right)\times\sigma^2\end{array}\right\}\\ Cov(X_{t+1},X_t)=Cov(a+bZ_{t+1}+cZ_{t-1},a+bZ_t+cZ_{t-2})=0\\ Cov(X_{t+2},X_t)=Cov(a+bZ_{t+2}+cZ_t,a+bZ_t+cZ_{t-2})=b\times c\times\sigma^2\\ \Longrightarrow \gamma_x(t+h\ ,\ t)=\begin{cases}(b^2+c^2)\sigma^2 \ , & \text{if } h=0\\ b\times c\times\sigma^2 & , \text{if } h\,\epsilon\,\{-2;+2\}\\ 0 & , else\end{cases}\end{array}\right\}\Longrightarrow \{X_t\} \text{ stationary}$$

b)

$$\left.\begin{array}{l}E[X_t]=0 \text{ independent of } t\\ V(X_t)=(\cos^2(ct)+\sin^2(ct))\times\sigma^2=\sigma^2\end{array}\right\}\Rightarrow E\left[X_t^2\right]=\sigma^2<\infty$$

$$Cov(X_{t+1},X_t)=Cov(Z_{t+1}\cos(ct)+Z_t\sin(ct),Z_t\cos(ct)+Z_{t-1}\sin(ct))=\sin(ct)\times\cos(ct)\times\sigma^2$$

$$\Rightarrow\gamma_x(t+h,t)=\begin{cases}\sigma^2 & , \text{if } h=0\\ \sin(ct)\times\cos(ct)\times\sigma^2 & , \text{if } h\,\epsilon\,\{-1,+1\} \quad \text{depends on } t\\ 0 & , else\end{cases}$$

$$\Longrightarrow \{X_t\} \quad \underline{\text{not}} \quad \text{stationary}$$

c)

$$\left.\begin{array}{l}\left.\begin{array}{l}E[X_t]=a\\ V(X_t)=b^2\sigma^2\end{array}\right\}\Rightarrow E\left[X_t^2\right]=a^2+b^2\sigma^2<\infty\\ Cov(X_{t+h},X_t)=Cov(a+bZ_0,a+bZ_0)=b^2\sigma^2\\ \qquad\Rightarrow\gamma_x(t+h,t)=b^2\sigma^2 \quad \forall h\epsilon Z.\end{array}\right\}\Longrightarrow\{X_t\} \text{ stationary}$$

d)

$$\left.\begin{array}{l}E[X_t]=E[Z_t]\times E[Z_{t-1}]=0\\ E\left[X_t^2\right]=E\left[Z_t^2\right]\times E\left[Z_{t-1}^2\right]=\sigma^2\times\sigma^2=\sigma^4<\infty\end{array}\right\}\Rightarrow V(X_t)=\sigma^4$$

$$Cov(X_{t+1},X_t)=Cov(Z_{t+1}Z_t,Z_t,Z_{t-1})=E\left[Z_{t+1}\times Z_t^2\times Z_{t-1}\right]-E[Z_{t+1}Z_t]\times E[Z_tZ_{t-1}]$$
$$=0$$

$$\Rightarrow\gamma_x(t+h,t)=\begin{cases}\sigma^4, & \text{if } h=0\\ 0 & , else\end{cases}\Longrightarrow\{X_t\} \quad \text{stationary}$$

2) *a)*

$$\gamma_x(t+h,t)=\begin{cases}(1+\theta^2)\cdot\sigma^2 & , \text{if } h=0\\ \theta\cdot\sigma^2 & , \text{if } h\,\epsilon\,\{-2,+2\}\\ 0 & , else\end{cases}=\begin{cases}1.64 \ , & h=0\\ 0.8 \ , & h\,\epsilon\,\{-2,+2\}\\ 0 \ , & else\end{cases}$$

$$\rho_x(h)=\begin{cases}1 & , it\ h=0\\ \theta/(1+\theta^2), & it\ h\in\{-2,2\}\\ 0 & , else\end{cases}=\begin{cases}1 & , h=0\\ 0.48780, & h\in\{-2,+2\}\\ 0 & , else\end{cases}$$

b) $V(\bar{X}_4)=\frac{1}{16}\times V(X_1+X_2+X_3+X_4)=\frac{1}{16}\times\left(\sum_{j=1}^{4}V(X_j)+2Cov(X_1,X_3)+2Cov(X_2,X_4)\right)$

$$=\frac{1}{4}\times1.64+\frac{1}{8}\times2\times0.8=0.61\searrow$$

symmetric around 0.41

c) $V(\bar{X}_4)=\frac{1}{4}\times1.64\times\frac{1}{8}\times2\times(-0.8)=0.21$ $\nearrow$

3)

Homework

4) *a)*

$$\sum_{j=-q}^{q} \tfrac{1}{2q+1} m_{t-j} = \sum_{j=-q}^{q} \tfrac{1}{2q+1} \left(c_0 + c_1(t-j)\right) = \tfrac{1}{2q+1} \sum_{j=-q}^{q} c_0 + \tfrac{1}{2q+1} \sum_{j=-q}^{q} c_1(t-j)$$

$$= \tfrac{1}{2q+1} \left(2q+1\right) c_0 + \tfrac{1}{2q+1} c_1 \left(2q+1\right) t - \tfrac{1}{2q+1} c_1 \sum_{j=-q}^{q} j$$

$$= m_t - \tfrac{c_1}{2q+1} \underbrace{\sum_{j=-q}^{q} j}_{=0} = m_t$$

b)

$$E\left[A_t\right] = E\left[\sum_{j=-q}^{q} \tfrac{1}{2q+1} Z_{t-j}\right] = 0$$

$$V\left(A_t\right) = V\left(\sum_{j=-q}^{q} \tfrac{1}{2q+1} Z_{t-j}\right) = \left(\tfrac{1}{2q+1}\right)^2 \sum_{j=-q}^{q} V\left(Z_{t-j}\right) = \left(\tfrac{1}{2q+1}\right)^2 \sigma^2 \sum_{j=-q}^{q} 1 = \tfrac{\sigma^2}{2q+1} \to 0\ ,\ as\quad q \to \infty$$

5)

$$X_t = a + bt + s_t + Y_t \quad , \quad E\left[Y_t\right] = 0 \quad , \quad s_t = s_{t-12} \quad , \quad \{Y_t\} \text{ is stationary}$$

$$\to \Delta_{12} X_t = (1 - L^{12}) X_t = X_t - X_{t-12} = \left(a + bt + s_t + Y_t\right) - \left(a + b(t-12) + s_{t-12} + Y_{(t-12)}\right)$$

$$= 12b + \left(Y_t - Y_{t-12}\right)$$

$$\to \underline{\Delta\Delta_{12} X_t} = \Delta\left(-12b + \left(Y_t - Y_{t-12}\right)\right) = (1-L)\left(12b + \left(Y_t - Y_{t-12}\right)\right)$$

$$= -12b + \left(Y_t - Y_{t-12}\right) + 12b - Y_{t-1} + Y_{t-13} = \underline{\left(Y_t - Y_{t-1}\right) - \left(Y_{t-12} - Y_{t-13}\right)}.$$

(Also with: $(1-L)\left(1-L^{12}\right) = \left(1 - L - L^{12} + L^{13}\right)$ applied to X_t)

Mean:

$$E\left[\underbrace{\left(Y_t - Y_{t-1}\right) - \left(Y_{t-12} - Y_{t-13}\right)}_{=W_t}\right] = 0$$

Autocovariance function:

$$\gamma_W(h) = Cov(W_{t+h}, W_t)$$

$$= Cov(Y_{t+h} - Y_{t+h-1} - Y_{t+h-12} + Y_{t+h-13}, Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13})$$

$$\underset{\text{linearity}}{\uparrow}\!\!= \gamma_Y(h+13) - 2\gamma_Y(h+12) + \gamma_Y(h+11) - 2\gamma_Y(h+1) + 4\gamma_Y(h) - 2\gamma_Y(h-1) + \gamma_Y(h-11)$$

$$-2\gamma_Y(h-12) + \gamma_Y(h-13)$$

6)

$$a(L) = a_0 + a_1 L + a_2 L^2 + \ldots\ldots$$

$$g(L) = g_0 + g_1 L + g_2 L^2 + \ldots..$$

$$\to L g(L) = g_0 L + g_1 L^2 + g_2 L^3 + \ldots\ldots.$$

$$\Longrightarrow a(1) + (1-L) g\left(L\right) = a(1) + g_0 + (g_1 - g_0) L + (g_2 - g_1) L^2 + \ldots..$$

Now: for $a(L) = a(1) + (1-L) g(L)$ all the coefficients in the expansion must be the same:

$$a_0 = a(1) + g_0; \qquad a_1 = g_1 - g_0; \qquad a_2 = g_2 - g_1; \qquad \ldots.$$

Let us now compute the derivative $\frac{d}{dL} a(L) = \frac{d}{dL}\left[a(1) + (1-L) g(L)\right] = -g(L) + (1-L) \frac{d}{dL} g(L)$

$\to$ let us evaluate in $\underline{1}$: $\frac{d}{dL} a(1) = -g(1)$

and with $\frac{d}{dL} a(1) = \sum_{j=0}^{\infty} j a_j$ we have $g(1) = -\sum_{j=0}^{\infty} j a_j$.